Universität Hamburg
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GrHyDy2024: Random spatial models
IMT Nord Europe, Lille

Sharp noise stability in continuum percolations via Spectra of Poisson functionals

Joint with G. Peccati (Luxembourg) and D. Yogeshwaran (ISI Bangalore)

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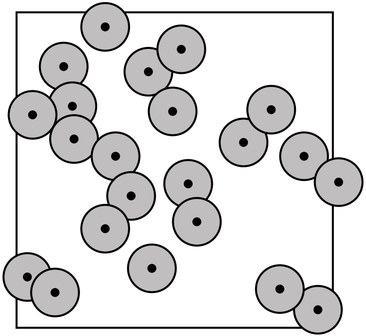
25.10.2024

Boolean percolation

Let η be a homogeneous Poisson process with intensity $\lambda > 0$ in $\mathbb{R}^2$. The occupied region in Boolean percolation is given by $\cup_{x \in \eta} B_1(x)$.



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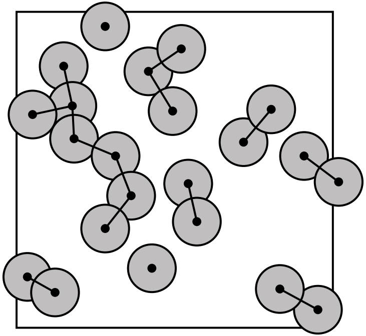

Naturally gives rise to a random geometric graph.

Left-right crossing of squares

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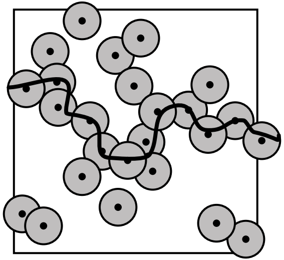
Let $f_L = \pm 1$ indicator of a L-R occupied crossing of $W_L = [-L, L]^2$:



Noise stability

To study the sensitivity of a system to random noise, one can measure its stability. For η the original configuration, let η^ε be its ε-noisy version.

Definition: $(f_L)_{L>0}$ is noise stable if

$$\lim_{\varepsilon \to 0} \sup_L \mathbf{P}\left\{ f_L(\eta) \neq f_L(\eta^\varepsilon) \right\} = 0.$$

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We consider the Ornstein-Uhlenbeck dynamics to perturb the system.

Ornstein Uhlenbeck dynamics

Given marked configuration η, noisy version η^ε is given by

$$\eta^\varepsilon = \eta_1 + \eta_2, \quad t > 0,$$

- η_1 a thinning of η [delete each point independently w.p. $(1 - e^{-\varepsilon})$],
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Motivation: Show sharp results for the OU dynamics.

Sharp noise instability in Boolean percolation (B., Peccati, Yogeshwaran '24+)

There exists $A_L \to \infty$ such that under the OU dynamics, $(f_L)_{L>0}$ exhibits **sharp noise instability** at time-scale $1/A_L$, i.e., for $\varepsilon_L A_L \to 0$,

$$\lim_{L\to\infty} \mathbf{P}\left\{f_L(\eta) \neq f_L(\eta^{\varepsilon_L})\right\} = 0,$$

while if $\varepsilon_L A_L \to \infty$,

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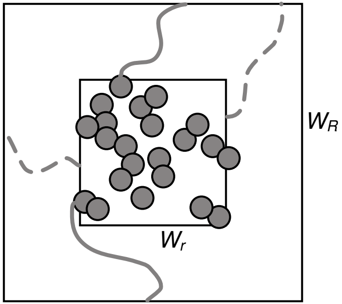
Here, one can take $A_L = L^2 \alpha_4(1, L) \to \infty$, where α_4 is the 4-arm probability.

Recall $A_L = L^2\alpha_4(1, L)$: For $1 \leq r \leq R < \infty$, $\alpha_4(r, R)$ is the probability of the event



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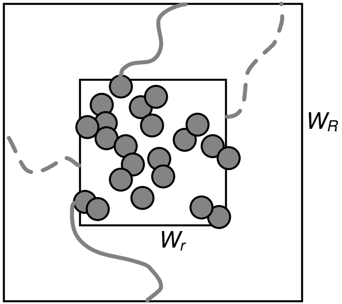


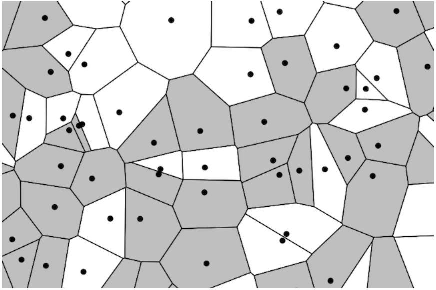
Quasi-multipicativity of α_4 (B., Peccati, Yogeshwaran '24+)

In critical Boolean percolation,

$$\alpha_4(r_1, r_3) \asymp \alpha_4(r_1, r_2)\alpha_4(r_2, r_3), \quad 1 \leq r_1 \leq r_2 \leq r_3.$$

Voronoi percolation

Given a homogeneous *Poisson process* with intensity 1 on $\mathbb{R}^2$, independently, colour each cell black or white with probabilities p and $1 - p$ (marked process η).



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By symmetry

$$\begin{aligned}\mathbf{P}_{1/2}(f_L(\eta) = 1) &= \mathbf{P}_{1/2}(\text{L-R black crossing of } W_L) \\ &= \mathbf{P}_{1/2}(\text{Top-Bottom white crossing of } W_L) = \frac{1}{2}.\end{aligned}$$

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$$\mathbf{P}_{1/2}(f_L(\eta^\varepsilon) = 1 | f_L(\eta)) \approx \frac{1}{2}.$$

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Definition: $(f_L)_{L>0}$ is noise sensitive (NS) under the dynamics η^ε if $\forall\, \varepsilon > 0$,

$$\lim_{L\to\infty} \mathrm{Cov}(f_L(\eta), f_L(\eta^\varepsilon)) = 0.$$

Frozen dynamics

- Sample the unperturbed random configuration according to $\eta \subset \mathbb{R}^2 \times \{\pm 1\}$.
- Resample the colour of each cell independently, at rate 1 to obtain the ε-noisy version η^ε after time $\varepsilon > 0$.

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Sharp NS under frozen dynamics (Vanneuville '21)

For the critical Voronoi percolation, $L^2\alpha_4(L) \to \infty$ and under the frozen dynamics,

$$\lim_{L\to\infty} \mathrm{Cov}(f_L(\eta), f_L(\eta^{\varepsilon_L})) = 0 \quad \text{when } \varepsilon_L L^2 \alpha_4(L) \to \infty,$$

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Only non-sharp results by Ahlberg, Baldasso '18; Last, Peccati, Yogeshwaran '23 under the OU dynamics.

Sharp NS under OU dynamics (B., Peccati, Yogeswaran '24+)

Under the OU dynamics, $(f_L)_{L>0}$ exhibits **sharp noise sensitivity** (also *sharp noise instability*) at scale $1/L^2\alpha_4(L)$, i.e.,

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For the proof of no noise sensitivity, we study the associated spectral point process.

Main tool : Concentration of spectral mass

Sensitivity to noise of a Boolean function is naturally encoded in its Fourier spectrum : *High frequency* implies sensitivity, while *low frequency* implies stability.

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The crossing functional $f_L(\eta)$ admits a unique *chaotic decomposition*

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For N_L with $\mathbf{P}\{N_L = k\} = k!\|u_k\|^2$, sharp noise instability follows by showing concentration of N_L around $\mathbf{E}N_L \asymp L^2\alpha_4(1, L)$.

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Given $N_L = k$, obtain the random vector $(X_1^{(k)}, \ldots, X_k^{(k)}) \in \mathbb{X}^k$ following a probability density proportional to

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Define the spectral process as

$$\boxed{\gamma_L(B) := \sum_{i=1}^{N_L} \delta_{X_i^{(N_L)}}(B), \quad B \in \mathcal{X}.}$$

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The size of the spectral process can be neatly related to the number of Pivotal points, i.e., those $x \in \eta$ such that $D_x^- f_L(\eta) \neq 0$.

Further remarks

Our results imply asymptotic Volatility (infinitely many jumps) of the crossing functional f_L over time intervals of length $t_L \gg (L^2\alpha_4(1, L))^{-1} \to 0$. Also has implications in critical windows for phase transition of f_L.

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It is typically easier to prove sharp noise stability. Showing sharp noise sensitivity under OU dynamics requires a precise understanding the lower tail of N_L : to conclude noise sensitivity when $\varepsilon_L \mathbf{E} N_L \to \infty$, it suffices to show that

$$\lim_{c\to 0} \liminf_{L\to\infty} \mathbf{P}\{N_L \geq c\,\mathbf{E} N_L\} = 1.$$

Thank you!

Pivotal point process

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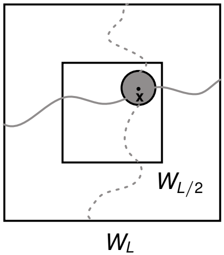
One can show for all $B \in \mathcal{X}$,

$$\mathbf{E}\left[|\gamma_L(B)|\right] = 4\mathbf{E}\left[|\mathcal{P}_L(B)|\right] = 4 \int_B \mathbf{P}\{D_x f_L(\eta) \neq 0\} \lambda(\mathrm{d}x).$$

Moment estimations

$$\mathbf{E}\left[|\gamma_L(B)|\right] = 4\int_B \mathbf{P}\{D_x f_L \neq 0\}\lambda(\mathrm{d}x).$$

In particular, $\mathbf{E}N_L = \mathbf{E}|\gamma_L| \geq \mathbf{E}|\gamma_L(W_{L/2})| \asymp L^2\alpha_4(1, L/2) \asymp L^2\alpha_4(1, L)$.



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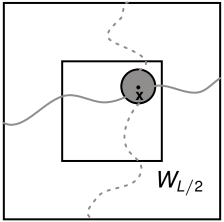

Indeed, $\mathbf{E}N_L \asymp L^2\alpha_4(1, L)$. A more careful second moment estimate then yields a desired concentration result in the Boolean model.

Voronoi: a covariance comparison inequality

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Noise sensitivity in the OU dynamics, when $\varepsilon_L L^2 \alpha_4(1, L) \to \infty$, now follows from the same for the frozen dynamics (Vanneuville '21).

Voronoi: no noise sensitivity when $\varepsilon_L L^2 \alpha_4(1, L) \to 0$

Let $f_L = \sum_{k=1}^{\infty} I_k(u_k)$. By Mehler's formula,

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Since $\mathbf{E} N_L \asymp L^2 \alpha_4(1, L)$, it follows that f_L is not noise sensitive when $\varepsilon_L L^2 \alpha_4(1, L) \to 0$.