Mean field stable matching

Mia Deijfen
Stockholm University

Joint with Daniel Ahlberg
Matteo Sfragara

Stable matchings

Gale & Shapley 1962:
"College admissions and stability of marriage"

Two sets of objects (men/women).

Preferences.

Matching stable if $\nexists$ pair that prefer each other before their current partners.

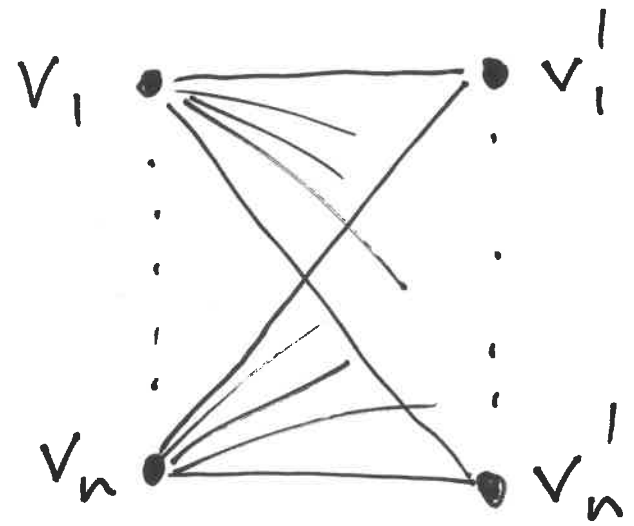
Our setting

Complete bipartite graph.

Vertex sets $\{v_i\}$ and $\{v_i'\}$, $i = 1, \ldots, n$.

Edge set $E = \{e = (v, v')\}$.

Edge costs $\{w(e)\}_{e \in E}$ iid $\mathrm{Exp}(1)$.



Matching: Subset $M \subset E$ of non-adjacent edges.

$c(v)$ = matching cost of v (∞ if v unmatched)

Stability:

$$\forall v, v' : (v, v') \notin M \Rightarrow w(v, v') > \min\{c(v), c(v')\}$$

Greedy algorithm

Select cheapest edge (v, v'). Erase all other edges incident to v and v'.

Iterate on remaining edges.

Gives unique stable matching S_n.

Total cost C_n.

Random assignment problem

Same setting.

$\{v_i\}$ = machines

$\{v_i'\}$ = jobs

Minimize total cost.

Gives minimal matching (MM).

Aldous 2001.

Total cost

Th.

- $\dfrac{E[C_n]}{\log n} \longrightarrow 1$
- $\mathrm{Var}\, C_n \longrightarrow \pi^2/6$
- $C_n - \log n \xrightarrow{d}$ Gumbel

Proof. $C_n = \sum_k \mathrm{Exp}(k)$.

MM: $E[C_n] \longrightarrow \pi^2/6$

Typical cost/vertex

Th. $n c(v) \xrightarrow{d} W$

$$f_W(x) = \frac{1}{(1+x)^2}, \quad x \geq 0.$$

MM: Exponentially decaying density ($\to 1/2$ as $x \to 0$).

Edge rank

Fix vertex v.

$R_n = r$ if matching uses r:th cheapest edge.

Th. $R_n \xrightarrow{d} R$

- $P(R=1) \approx 0.596$
- $P(R>r) \sim \frac{1}{r}$

Proof. Transfer computation to limiting object (PWIT).

MM: $P(R=r) = 2^{-r}$.

Perturbed configuration

$\{w(e)\}$ and $\{\tilde{w}(e)\}$ indep. iid $\mathrm{Exp}(1)$.

$\{U(e)\}$ iid unif$(0,1)$.

$$w_\varepsilon(e) = \begin{cases} w(e) & \text{if } U(e) > \varepsilon \\ \tilde{w}(e) & \text{if } U(e) \leq \varepsilon \end{cases}$$

Stable matching S_n^ε.

Total cost C_n^ε.

Robustness of matching

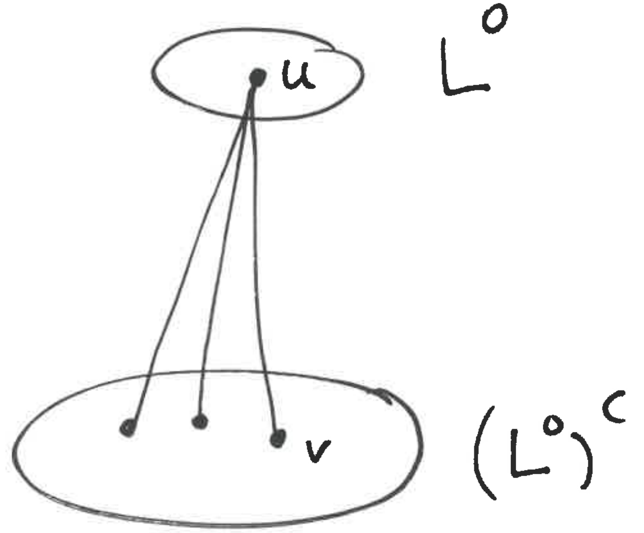
Th. $$\lim_n \frac{E|S_n^0 \cap S_n^\varepsilon|}{n} \geq 1 - \underbrace{f(\varepsilon)}_{\to 0 \text{ as } \varepsilon \to 0}.$$

Proof. Descending paths.

Sensitivity of last edges

$L_m^\varepsilon = \{m \text{ most expensive edges in } S_n^\varepsilon\}$

Th. $m \ll \varepsilon \log n$: $L_m^0 \cap L_m^\varepsilon = \emptyset$ whp.



Proof. Each vertex $u \in L^0$ desired by many vertices $v \in (L^0)^c$ after resampling.

Sensitivity of total cost

Th. $\varepsilon \log n \gg 1$: $\mathrm{Corr}(C_n^0, C_n^\varepsilon) \longrightarrow 0$.

$m \ll n$:

- $E\left[\begin{matrix}\text{cost of } n-m \\ \text{first edges}\end{matrix}\right] \sim \log\left(\frac{n}{m}\right)$
- $\mathrm{Var}\left(\begin{matrix}\text{cost of } n-m \\ \text{first edges}\end{matrix}\right) \sim \frac{1}{m}$

Further work

- Show $\mathrm{Corr}(C_n^0, C_n^\varepsilon) \not\to 0$ for $\varepsilon \log n \ll 1$.
- Other cost distributions.