Ordering and Convergence of Large Degrees in Random Hyperbolic Graphs

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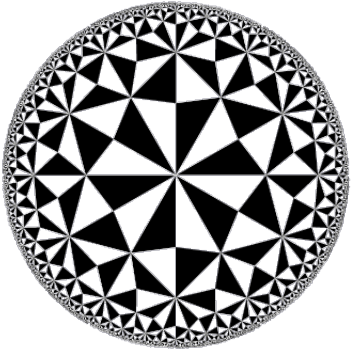
Hyperbolic geometry

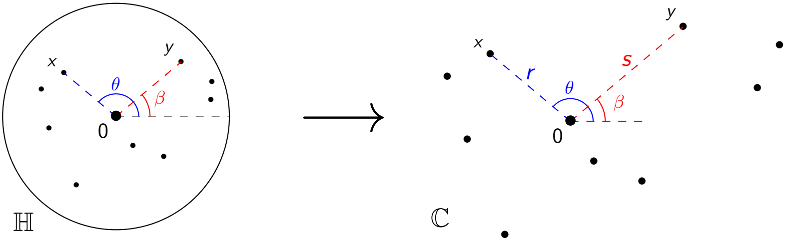
Poincaré disc $\mathbb{H}$: unit disc of $\mathbb{C}$ equipped with

$$\mathbf{g}_{\mathbb{H}} := \frac{4\mathbf{g}_{\mathbb{C}}}{(1-|w|^2)^2}, \quad \text{where } \mathbf{g}_{\mathbb{C}} \text{ is the Euclidean metric on } \mathbb{C}$$

$\rightarrow d_{\mathbb{H}}$ Riemannian distance



Coordinates and native representation



For $x = (r:\theta)$ and $y = (s:\beta)$ two points of $\mathbb{H}$

$$\cosh(d_{\mathbb{H}}(x,y)) = \cosh(r)\cosh(s) - \sinh(r)\sinh(s)\cos(\theta-\beta)\,.$$

$\mathcal{B}_x(s)$ open ball for distance $d_{\mathbb{H}}$

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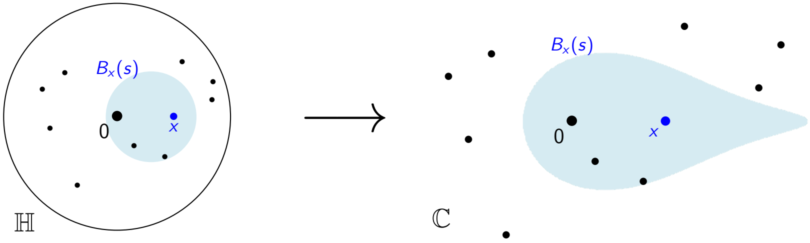

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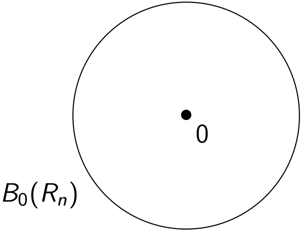

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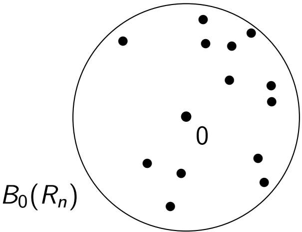
Measure μ_n on $\mathcal{B}_0(R_n)$ with radial density:

$$\rho_n(r) := \frac{\alpha \sinh(\alpha r)}{(\cosh(\alpha R_n) - 1)} \mathbf{1}_{\{r<R_n\}}$$

$(X_1, X_2, \ldots, X_n)$ i.i.d with distribution μ_n



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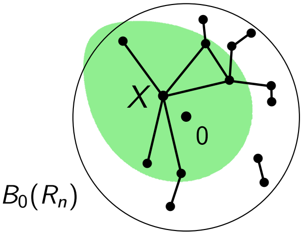

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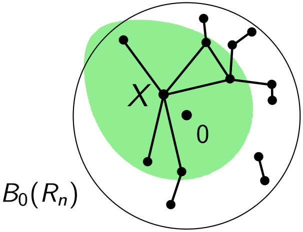
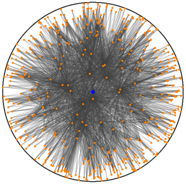


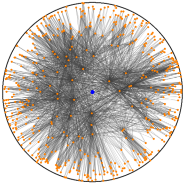
$\deg(X) =$ number of neighbours of X

$\mathbb{E}_X[\deg(X)] = (n-1)\mu_n(\mathcal{B}_X(R_n)) \searrow$ in $r(X)$

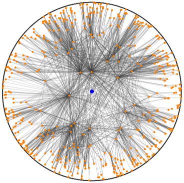
Three different regimes (see [BFM16])



$\alpha < 1/2$, dense regime, connected with high probability (hubs near the centre)



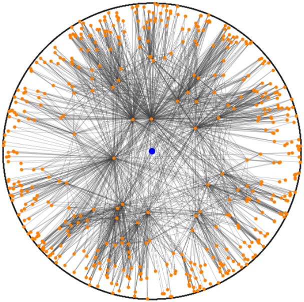
$\alpha = 1/2$, probability of connection converges to a constant



$\alpha > 1/2$, disconnected with high probability

For $\alpha > 1/2$, model for complex networks [AB02]:

- sparseness [Pet14]
- small world [ABF17]
- high clustering [CF16, FvdHMS21, GPP12]
- **scale-free degree distribution**
  - [GPP12] Random hyperbolic graphs: degree sequence and clustering



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Theorem: [GPP12] Power-law with exponent $2\alpha + 1$

For $\alpha > 1/2$ and $v_n \to \infty$, w.h.p., the maximum degree belongs to

$$[n^{1/(2\alpha)} v_n^{-1}, n^{1/(2\alpha)} v_n]$$

For $n^\delta \le k \le \frac{n^{1/(2\alpha)}}{\log(n)}$, w.h.p., the fraction of vertices of degree $\ge k$ is

$$(1 + o(1)) C_{\alpha,\nu} k^{-2\alpha}$$

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Better estimate on maximum/large degrees? For $\alpha \leq 1/2$?

Notations

Point process of the degrees:

$$\mathcal{D}_n := \sum_{i=1}^{n} \delta_{\deg(X_i)}$$

$r(X)$ radius of the node X

$X_{(1)}, X_{(2)}, \ldots, X_{(n)}$ ordering by increasing radius

Ordering and convergence of maximum degrees

Theorem: [LG24] Maximum degree in RHGs

For $\alpha > 0$ and k fixed, with high probability,

$$\deg(X_{(1)}) > \deg(X_{(2)}) > \cdots > \deg(X_{(k)}) > \deg(X_{(i)}), \quad \forall i > k$$

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Moreover, for $\alpha < 1/2$,

$$\mathcal{D}_n(n - n^{\alpha+1/2} \cdot\,) \xrightarrow[n\to\infty]{(d)} \eta_{m_1}, \quad \text{in } M_p([0,\infty))$$

For $\alpha = 1/2$,

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In particular, for $\alpha < 1/2$,

$$n^{-(\alpha+1/2)}(n - D_n^{\max}) \xrightarrow[n\to\infty]{(d)} \text{Weibull}\left(2, \pi^{-1}\nu^{-\alpha}\right)$$

For $\alpha = 1/2$,

$$n^{-1} D_n^{\max} \xrightarrow[n\to\infty]{(d)} V(2\,\text{arcosh}(\text{Exponential}\,(\nu) + 1))$$

For $\alpha > 1/2$,

$$n^{-\frac{1}{2\alpha}} D_n^{\max} \xrightarrow[n\to\infty]{(d)} \text{Fréchet}\,(2\alpha, C_\alpha \nu)$$

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Point process of the node radii, $\mathcal{R}_n := \sum_{i=1}^{n} \delta_{r(X_i)}$

$$\mathcal{R}_n((1 - \tfrac{1}{2\alpha})R_n + \ \cdot\ ) \xrightarrow[n\to\infty]{(d)} \eta_{m_6}, \quad \text{in } M_p([0,\infty))$$

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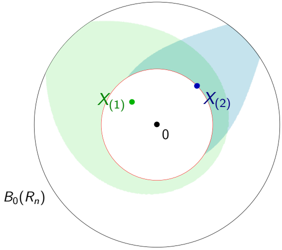

Estimate of [GPP12]

$$\mu_n\left(\mathcal{B}_x(R_n)\right) = C_\alpha e^{-r(x)/2}\left(1 + O(e^{-(\alpha-1/2)r(x)} + e^{-r(x)})\right)$$

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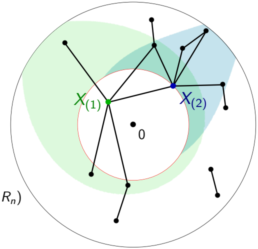

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By Chernoff bounds

$$\mathbb{P}\left[\deg(X_{(1)}) \leq \deg(X_{(2)})\right] \leq \exp\left(-n^{1/(2\alpha)+o(1)}\right)$$

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Moreover

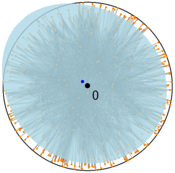
$$\mathcal{D}_n(n^{1/(2\alpha)} \cdot\,) \xrightarrow[n\to\infty]{(d)} \eta_{m_3}, \quad \text{in } M_p((0,\infty])$$

$(\deg(X_{(1)}), \ldots, \deg(X_{(k)}))$ concentrate on there conditional expectations

$$\mathbb{E}_{X_{(i)}}[\deg(X_{(i)})] \sim n\mu_n(\mathcal{B}_{X_{(i)}}(R_n)) \sim C_\alpha n e^{-r(X_{(i)})/2}$$

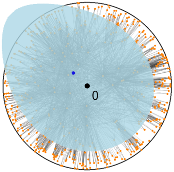
$$\mathcal{R}_n((1-\tfrac{1}{2\alpha})R_n + \,\cdot\,) \xrightarrow[n\to\infty]{(d)} \eta_{m_6} \quad \rightsquigarrow \quad \mathcal{D}_n(n^{1/(2\alpha)} \cdot\,) \xrightarrow[n\to\infty]{(d)} \eta_{m_3}$$

What about the other regimes in α?



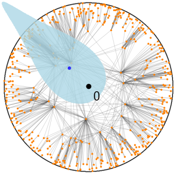
For $\alpha < 1/2$,

$$\mathcal{R}_n(n^{-\frac{1}{2}(1-2\alpha)}\ \cdot\ ) \xrightarrow[n\to\infty]{(d)} \eta_{m_4},\ \text{in } M_p([0,\infty))$$



For $\alpha = 1/2$,

$$\mathcal{R}_n(\ \cdot\ ) \xrightarrow[n\to\infty]{(d)} \eta_{m_5},\ \text{in } M_p([0,\infty))$$



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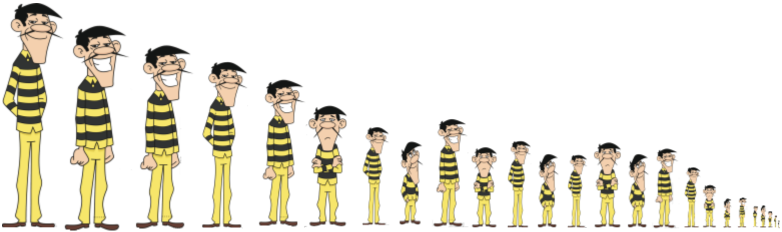
Ordering of the nodes?

$$\deg(X_{(1)}) > \deg(X_{(2)}) > \cdots > \deg(X_{(k)}) > \deg(X_{(i)}), \quad \forall i > k$$



Ordering up to a polynomial rank (scale free regime)

Let us fix $\alpha > \frac{7+\sqrt{33}}{16} \approx 0.8$ and $v_n \to \infty$. Define

$$\beta_c := \frac{1}{1+8\alpha} \quad \text{and} \quad k_n := n^{\beta_c} / \log(n)^{2\alpha}$$

Theorem: [LG24+]

For $\alpha > \frac{7+\sqrt{33}}{16}$, with high probability,

$$\deg(X_{(1)}) > \deg(X_{(2)}) > \cdots > \deg(X_{(k_n)}) > \deg(X_{(i)}), \ \forall i > k_n$$

and there exists $i \in [n^{\beta_c}, n^{\beta_c} v_n]$ such that

$$\deg(X_{(i)}) < \deg(X_{(i+1)})$$

Proof sketch: ordering up to k_n

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$W_k := n\mu_n\left(\mathcal{B}_{X_{(k)}}(R_n)\right) \sim C_\alpha n e^{-r(X_{(k)})/2}$

$\Delta_k := r(X_{(k+1)}) - r(X_{(k)})$

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Using Chernoff bounds, we get

$$\begin{aligned}\mathbb{P}_{X_{(k)},X_{(k+1)}}\left[\deg(X_{(k)}) \leq \deg(X_{(k+1)})\right] &\leq \exp\left(-C_1 W_k\left(1-\frac{W_{k+1}}{W_k}\right)^2\right)\\ &\leq \exp\left(-C_2 W_k \Delta_k^2\right)\end{aligned}$$

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$$\leq \exp\left(-C_2 W_k \Delta_k^2\right)$$

$\rightsquigarrow$ choose k_n such that w.h.p., for all $k \leq k_n$,

$$\exp\left(-C_2 W_k \Delta_k^2\right) = o(1/n)$$

By a union bound,

$$\mathbb{P}\left[\deg(X_{(1)}) > \deg(X_{(2)}) > \cdots > \deg(X_{(k_n)})\right] = 1 - o(k_n/n) - o(1)$$

Proof sketch: no ordering after n^{β_c}

Theorem: [LG24+]

W.h.p., $\exists k \in [n^{\beta_c}, n^{\beta_c} v_n],\ \deg(X_{(k)}) < \deg(X_{(k+1)})$

Reminder: $W_k := n\mu_n\left(\mathcal{B}_{X_{(k)}}(R_n)\right), \quad \Delta_k := r(X_{(k+1)}) - r(X_{(k)})$

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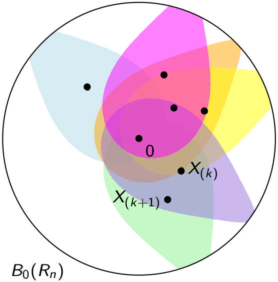

W.h.p, more than $C_1 v_n$ indices $k \in [n^{\beta_c}, n^{\beta_c} v_n]$ satisfy

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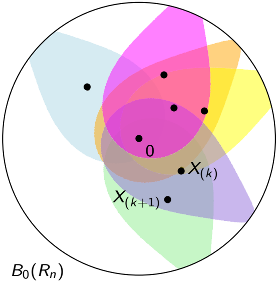

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$$W_k \Delta_k^2 \leq \delta_1.$$

This can be rewritten

$$W_k - W_{k+1} \leq \delta_2 \operatorname{Var}_{X_{(k+1)}}(\deg(X_{(k+1)})),$$

which implies

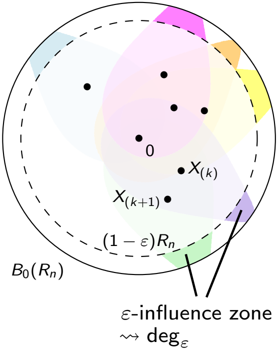
$$\mathbb{P}_{X_{(k)}, X_{(k+1)}}\left[\deg(X_{(k+1)}) > W_k\right] \geq \delta_3.$$

Proof sketch: no ordering after n^{β_c}

Theorem: [LG24+]

W.h.p., $\exists k \in [n^{\beta_c}, n^{\beta_c} v_n],\ \deg(X_{(k)}) < \deg(X_{(k+1)})$

Reminder: $W_k := n\mu_n\left(\mathcal{B}_{X_{(k)}}(R_n)\right), \quad \Delta_k := r(X_{(k+1)}) - r(X_{(k)})$



W.h.p, we can find $C_2 v_n$ indices $k \in [n^{\beta_c}, n^{\beta_c} v_n]$ s.t.

$$\mathbb{P}_{X_{(k)}, X_{(k+1)}}\left[\deg_\varepsilon(X_{(k+1)}) > W_k^\varepsilon\right] \geq \delta$$

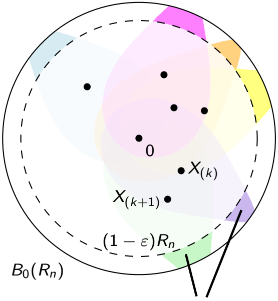
+ disjoint condition after $(1-\varepsilon)R_n$ (for all points)

Proof sketch: no ordering after n^{β_c}

Theorem: [LG24+]

W.h.p., $\exists k \in [n^{\beta_c}, n^{\beta_c} v_n]$, $\deg(X_{(k)}) < \deg(X_{(k+1)})$

Reminder: $W_k := n\mu_n\left(\mathcal{B}_{X_{(k)}}(R_n)\right)$, $\quad \Delta_k := r(X_{(k+1)}) - r(X_{(k)})$



W.h.p, we can find $C_2 v_n$ indices $k \in [n^{\beta_c}, n^{\beta_c} v_n]$ s.t.

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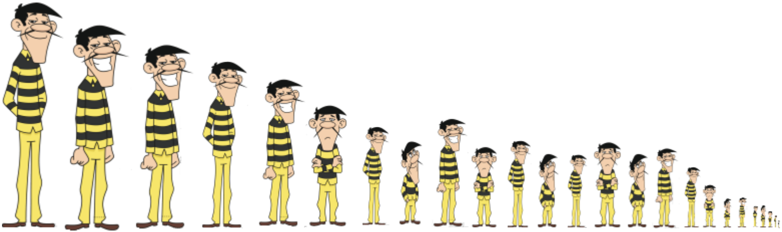
$+$ disjoint condition after $(1-\varepsilon)R_n$ (for all points)

$\rightsquigarrow$ W.h.p, there exists $k \in [n^{\beta_c}, n^{\beta_c} v_n]$, such that

$$\deg_\varepsilon(X_{(k)}) < \deg_\varepsilon(X_{(k+1)})$$

Thank you for your attention!



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