Quenched critical percolation on Galton–Watson trees

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Joint work with Quirin Vogel

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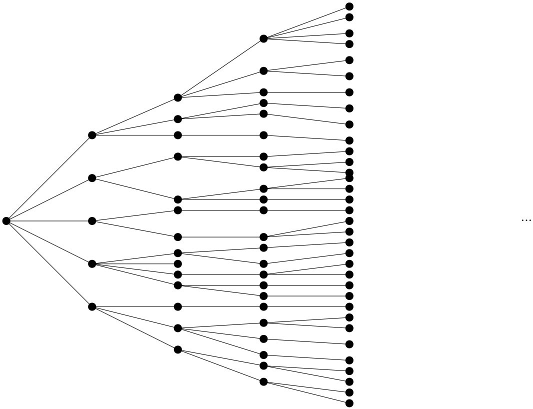
The model

Let T be a supercritical Galton–Watson tree with no leaves, with offspring law ξ. Let $\mu > 1$ be the mean number of offspring, and $\mathbf{o}$ the root.



(no leaves means almost sure survival - convenient)

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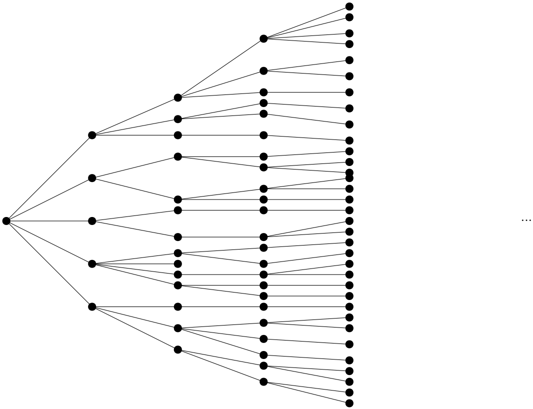

Important fact: let Z_n be the number of individuals at generation n. Then $\exists$ r.v. W, supported on $(0, \infty)$, such that, a.s., $\frac{Z_n}{\mu^n} \to W$.

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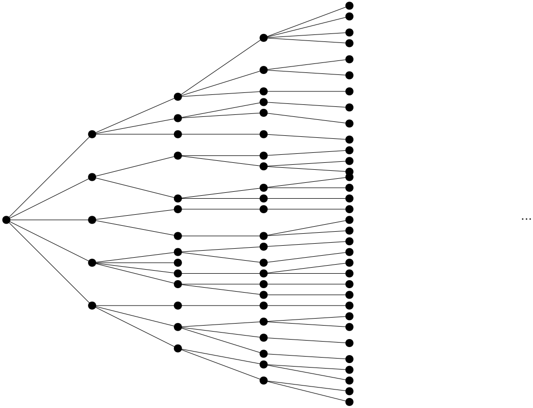

We consider **Bernoulli percolation** on T: fix $p \in (0, 1)$, and each edge is independently open with probability p, or closed with probability $1 - p$.

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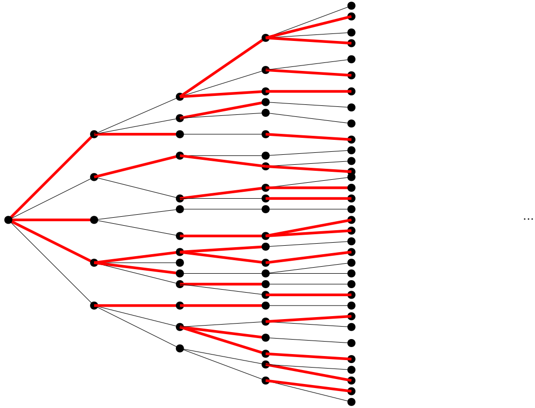

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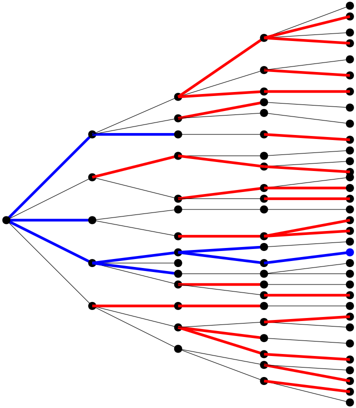

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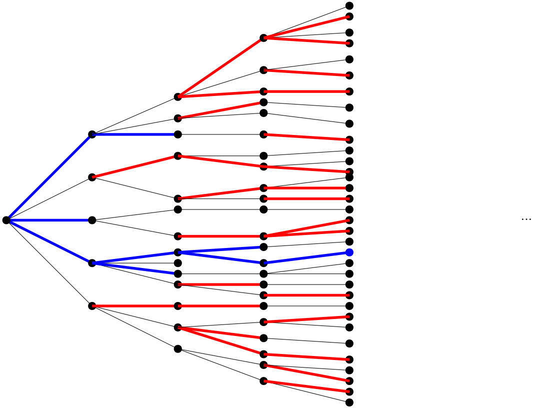
Percolation on T



As usual, we define:

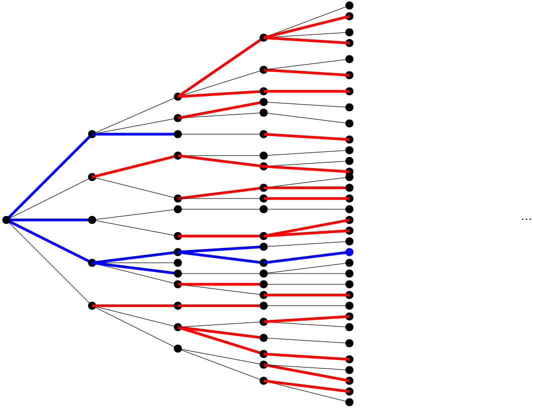
$$p_c = \inf\{p > 0 : \mathbb{P}\Big(\mathbf{o} \overset{p}{\longleftrightarrow} \infty\Big) > 0\}.$$

What is $p_c(T)$?

Let C denote the root cluster (the blue structure).

Observation: C has the law of a Galton-Watson tree.

Offspring law: first sample $N \sim \xi$, then take Binomial(N, p).



Hence: $\mathbb{P}\left(\mathbf{o} \overset{p}{\longleftrightarrow} \infty\right) > 0$ iff mean > 1, i.e. iff $\mathbb{E}[Np] > 1$.

Hence $p_c = 1/\mu$.

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Another point of view: for a given realisation of T, we can set

$$p_c(T) = \inf \left\{ p > 0 : \mathbb{P}_T\Big(\mathbf{o} \overset{p}{\longleftrightarrow} \infty\Big) > 0 \right\},$$

where $\mathbb{P}_T(\cdot)$ denotes the law of percolation on the explicit tree T.

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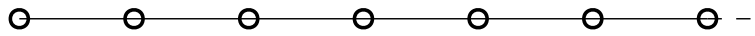
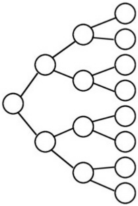
Question: is it true that $p_c(T) = 1/\mu$ for almost every realisation of T?

Example of a random tree where there is a difference

Consider the random tree $\tilde{T}$ which is equal to T_1, the 1-regular tree, with prob $1/2$, and T_2, the 2-regular tree, with prob $1/2$.





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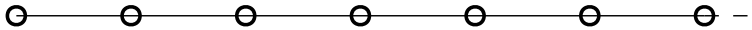

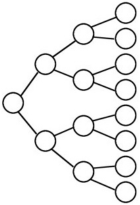

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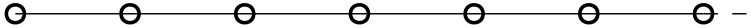

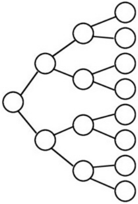

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Overall $p_c = 1/2$ since for $p < 1/2$,

$$\mathbb{P}\left(\mathbf{o} \overset{p}{\longleftrightarrow} \infty\right) = \frac{1}{2}\mathbb{P}_{T_2}\left(\mathbf{o} \overset{p}{\longleftrightarrow} \infty\right) + \frac{1}{2}\mathbb{P}_{T_1}\left(\mathbf{o} \overset{p}{\longleftrightarrow} \infty\right) = 0,$$

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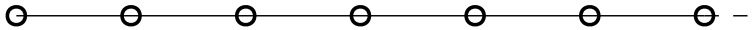

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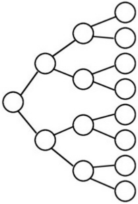

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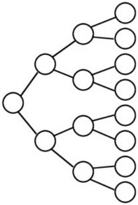

$p_c(T_2) = 1/2$

$p_c = 1/2$, but it is **not** true that $p_c(\tilde{T}) = 1/2$ almost surely.

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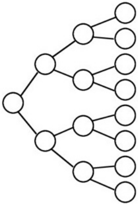

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$p_c = 1/2$, but it is **not** true that $p_c(\tilde{T}) = 1/2$ almost surely.

For our supercritical GW: $p_c = 1/\mu$, almost surely (Lyons 1990).

Quenched vs annealed results: notation

$T_n = n^{th}$ generation of T, $C =$ cluster of $\mathbf{o}$, $Y_n = |C \cap T_n|$,
$C_{\geq n} = C$ conditioned to have size n, $W = \lim \frac{Z_n}{m^n}$

$\mathbf{P} =$ law of T
$\mathbb{P}_T =$ law of percolation on T, given T
$\mathbb{P} = \mathbf{P} \times \mathbb{P}_T$, annealed law

Quenched vs annealed results - finite variance

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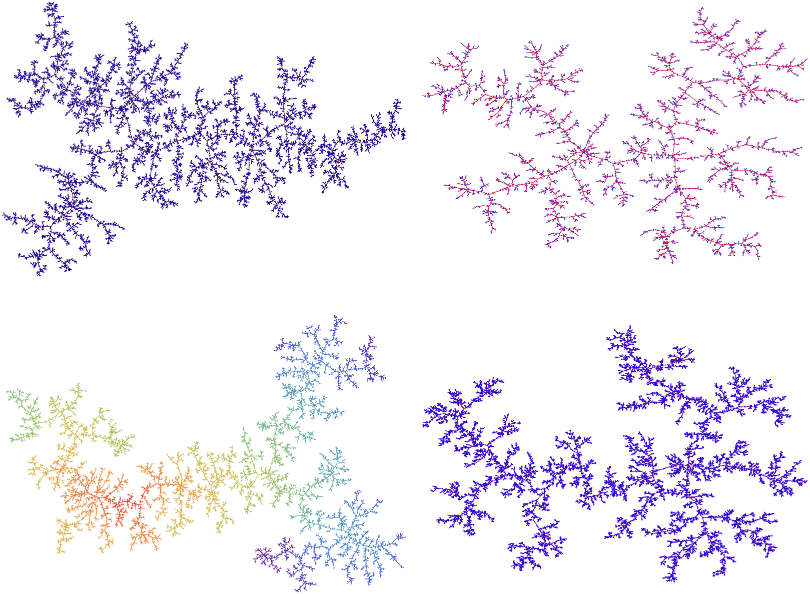
The CRT



Pictures by Igor Kortchemski and Laurent Ménard.

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Quenched vs annealed results - stable analogues

Stable tails on offspring law: $\xi(x,\infty) \sim cx^{-\alpha}$, where $\alpha \in (1,2)$.
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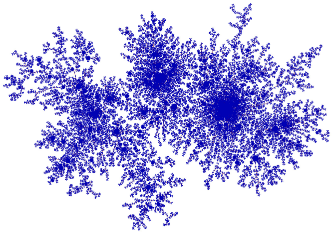
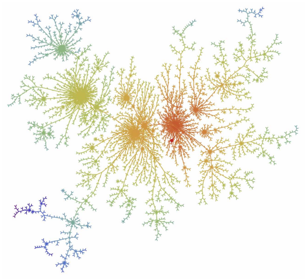
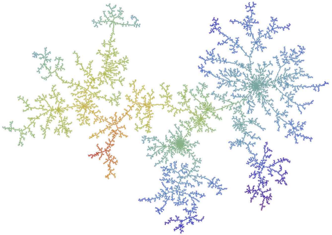
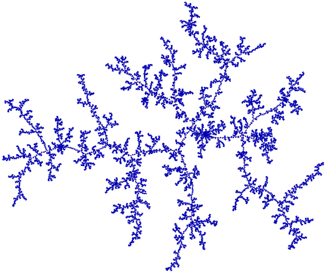
$p_c = 1/\mu$	$p_c(T) = 1/\mu$ a.s.
$\mathbb{P}\left(\mathbf{o} \overset{p_c}{\longleftrightarrow} T_n\right) \sim cn^{-\frac{1}{\alpha-1}}$	$\mathbb{P}_T\left(\mathbf{o} \overset{p_c}{\longleftrightarrow} T_n\right) \sim W \cdot cn^{-\frac{1}{\alpha-1}}$
$\mathbb{P}(\lvert C\rvert \geq n) \sim c' n^{-1/\alpha}$	$\mathbb{P}_T(\lvert C\rvert \geq n) \sim W \cdot c' n^{-1/\alpha}$
Given $Y_n > 0$: $n^{-\frac{1}{\alpha-1}} Y_n \overset{(d)}{\to} Y$	$n^{-\frac{1}{\alpha-1}} Y_n^T \overset{(d)}{\to} Y$ a.s.
$\left(n^{-\frac{1}{\alpha-1}} Y_{n(1+t)}\right)_{t \geq 0} \overset{(d)}{\to} (Y_t)_{t \geq 0}$	$\left(n^{-\frac{1}{\alpha-1}} Y^T_{n(1+t)}\right)_{t \geq 0} \overset{(d)}{\to} (Y_t)_{t \geq 0}$ a.s.
Given $Y_\infty > 0$: $n^{-\frac{1}{\alpha-1}} Y_n \overset{(d)}{\to} \hat{Y}$	$n^{-\frac{1}{\alpha-1}} Y_n^T \overset{(d)}{\to} \hat{Y}$
$(C_{\geq n}, n^{-(1-1/\alpha)} d_n, \frac{1}{n}\mu_n) \underset{\text{GHP}}{\overset{(d)}{\to}} \mathcal{T}_\alpha$	$(C^T_{\geq n}, n^{-(1-1/\alpha)} d_n, \frac{1}{n}\mu_n) \underset{\text{GHP}}{\overset{(d)}{\to}} \mathcal{T}_\alpha$ a.s.

Stable trees



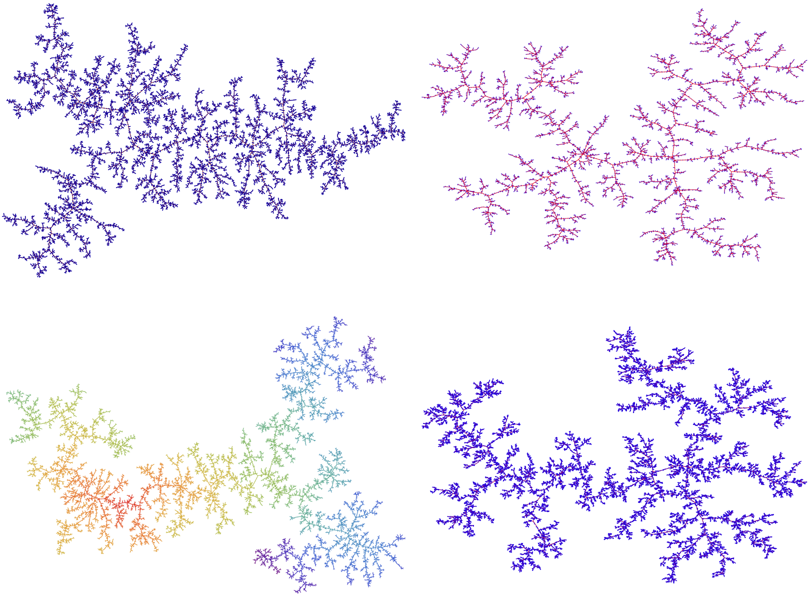






Pictures by Igor Kortchemski.

The CRT



Pictures by Igor Kortchemski and Laurent Ménard.

Connection probabilities: $\mathbb{P}_T\left(\mathbf{o} \xleftrightarrow{p_c} T_n\right)$

Choose $0 \ll m \ll n$ (in fact $m = C \log n$). Intuition:

$$\mathbb{P}_T\left(\mathbf{o} \xleftrightarrow{p_c} T_n\right) \approx \sum_{v \in T_m} \mathbb{P}_T\left(\mathbf{o} \leftrightarrow v \overset{*}{\leftrightarrow} T_n\right)$$

Error bounded by (use inclusion-exclusion):

$$\sum_{\substack{u,v \in T_m \\ u \neq v}} \mathbb{P}_T\left(\mathbf{o} \leftrightarrow (u, v) \overset{*}{\leftrightarrow} T_n\right).$$

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Natural strategy: use second moment and Borel-Cantelli. Show that

$$\mathbb{P}_T\left(\mathbf{o} \xleftrightarrow{p_c} T_n\right) - W \cdot cn^{-\frac{1}{\alpha-1}} = o(n^{-\frac{1}{\alpha-1}})$$

with probability at least $1 - n^{-2}$.

Connection probabilities: $\mathbb{P}_T\left(\mathbf{o} \overset{p_c}{\longleftrightarrow} T_n\right)$

Choose $0 \ll m \ll n$ (in fact $m = C \log n$). Intuition:

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with probability at least $1 - n^{-2}$.

Problem: This is asking for a lot. Can we get away with weaker tail decay?

Connection probabilities: $\mathbb{P}_T\left(\mathbf{o} \overset{p_c}{\longleftrightarrow} T_n\right)$

Want:

$$\mathbb{P}_T\left(\mathbf{o} \overset{p_c}{\longleftrightarrow} T_n\right) - W \cdot cn^{-\frac{1}{\alpha-1}} = o(n^{-\frac{1}{\alpha-1}}). \qquad (1)$$

Set $p_n = \mathbb{P}_T\left(\mathbf{o} \overset{p_c}{\longleftrightarrow} T_n\right)$. Note that p_n is decreasing.

Connection probabilities: $\mathbb{P}_T\left(\mathbf{o} \overset{p_c}{\longleftrightarrow} T_n\right)$

Want:

$$\mathbb{P}_T\left(\mathbf{o} \overset{p_c}{\longleftrightarrow} T_n\right) - W \cdot cn^{-\frac{1}{\alpha-1}} = o(n^{-\frac{1}{\alpha-1}}). \qquad (1)$$

Set $p_n = \mathbb{P}_T\left(\mathbf{o} \overset{p_c}{\longleftrightarrow} T_n\right)$. Note that p_n is decreasing.

Suppose we could prove (1) only for even n. Then, for odd n,

$$W \cdot cn^{-\frac{1}{\alpha-1}} \sim p_{n-1} \leq p_n \leq p_{n+1} \sim W \cdot cn^{-\frac{1}{\alpha-1}},$$

so we get the result for odd n for free.

How far can we push this?

Connection probabilities: $\mathbb{P}_T\left(\mathbf{o} \xleftrightarrow{p_c} T_n\right)$

In fact: it suffices to prove convergence along any sub-exponentially growing sequence. We set $n_k = k^K$, where K is large, and use Borel-Cantelli to show that

$$p_{n_k} \sim W \cdot c n_k^{-\frac{1}{\alpha-1}}$$

almost surely.

Connection probabilities: $\mathbb{P}_T\left(\mathbf{o} \overset{p_c}{\longleftrightarrow} T_n\right)$

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$$p_{n_k} \sim W \cdot c n_k^{-\frac{1}{\alpha-1}}$$

almost surely.

Then if $n \in [n_k, n_{k+1}]$:

$$n^{\frac{1}{\alpha-1}} p_n \leq \left(\frac{n_{k+1}}{n_k}\right)^{\frac{1}{\alpha-1}} n_k^{\frac{1}{\alpha-1}} p_{n_k} \sim cW,$$

and similarly for the lower bound, so we get that $p_n \sim W \cdot c n^{-\frac{1}{\alpha-1}}$.

Connection probabilities: $\mathbb{P}_T\left(\mathbf{o} \overset{p_c}{\longleftrightarrow} T_n\right)$

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and similarly for the lower bound, so we get that $p_n \sim W \cdot c n^{-\frac{1}{\alpha-1}}$.

Since K can be as large as we like, we just need to obtain tail decay of $n^{-\varepsilon}$ and then set $K = 2\varepsilon^{-1}$.

Connection probabilities: $\mathbb{P}_T\left(\mathbf{o} \overset{p_c}{\longleftrightarrow} T_n\right)$

Choose $0 \ll m \ll n$. Intuition:

$$\mathbb{P}_T(Y_n > 0) \approx \sum_{v \in T_m} \mathbb{P}_T\left(\mathbf{o} \leftrightarrow v \overset{*}{\leftrightarrow} T_n\right)$$

Error bounded by (use inclusion-exclusion):

$$\sum_{\substack{u,v \in T_m \\ u \neq v}} \mathbb{P}_T\left(\mathbf{o} \leftrightarrow (u,v) \overset{*}{\leftrightarrow} T_n\right).$$

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Strategy: show that

$$\mathbb{P}_T(\mathbf{o} \longleftrightarrow T_n) - W \cdot c n^{-\frac{1}{\alpha-1}} = o(n^{-\frac{1}{\alpha-1}})$$

with probability at least $1 - n^{-\varepsilon}$.

Connection probabilities: $\mathbb{P}_T\left(\mathbf{o} \stackrel{p_c}{\longleftrightarrow} T_n\right)$ - main term

Apply Chebyshev to $\sum_{v \in T_m} \mathbb{P}_T\left(\mathbf{o} \leftrightarrow v \stackrel{*}{\leftrightarrow} T_n\right)$.

Connection probabilities: $\mathbb{P}_T\left(\mathbf{o} \xleftrightarrow{p_c} T_n\right)$ - main term

Apply Chebyshev to $\sum_{v \in T_m} \mathbb{P}_T\left(\mathbf{o} \leftrightarrow v \overset{*}{\leftrightarrow} T_n\right)$.
Expectation:

$$\mathbf{E}\left[\sum_{v \in T_m} \mathbb{P}_T\left(\mathbf{o} \leftrightarrow v \overset{*}{\leftrightarrow} T_n\right) \,\middle|\, \mathcal{F}_m\right] = \frac{|T_m|}{\mu^m} \mathbb{P}(\mathbf{o} \leftrightarrow T_{n-m}) \sim Wcn^{-\frac{1}{\alpha-1}}.$$

Connection probabilities: $\mathbb{P}_T\left(\mathbf{o} \overset{p_c}{\longleftrightarrow} T_n\right)$ - main term

Apply Chebyshev to $\sum_{v\in T_m} \mathbb{P}_T\left(\mathbf{o} \leftrightarrow v \overset{*}{\longleftrightarrow} T_n\right)$.
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$$\mathbf{E}\left[\sum_{v\in T_m} \mathbb{P}_T\left(\mathbf{o} \leftrightarrow v \overset{*}{\longleftrightarrow} T_n\right) \,\middle|\, \mathcal{F}_m\right] = \frac{|T_m|}{\mu^m}\mathbb{P}(\mathbf{o} \leftrightarrow T_{n-m}) \sim Wcn^{-\frac{1}{\alpha-1}}.$$

Variance:

$$\begin{aligned}\mathbf{Var}\left(\sum_{v\in T_m} \mathbb{P}_T\left(0 \leftrightarrow v \overset{*}{\longleftrightarrow} T_n\right)\middle|\mathcal{F}_m\right) &= \mu^{-2m} \sum_{v\in T_m} \mathbf{Var}\left(\mathbb{P}_T\left(v \overset{*}{\longleftrightarrow} T_n\right)\right)\\ &\le \mu^{-2m} \sum_{v\in T_m} \mathbf{E}\left[\mathbb{P}_T\left(v \overset{*}{\longleftrightarrow} T_n\right)\right]\\ &\le c|T_m|n^{-\frac{1}{\alpha-1}}\mu^{-2m}.\end{aligned}$$

Connection probabilities: $\mathbb{P}_T\left(\mathbf{o} \xleftrightarrow{p_c} T_n\right)$ - main term

Expectation $\sim Wcn^{-\frac{1}{\alpha-1}}$
Variance $\leq C|T_m|n^{-\frac{1}{\alpha-1}}\mu^{-2m}$.

Need sum to be within $o(n^{-\frac{1}{\alpha-1}})$ of its mean, say within $n^{-\frac{1}{\alpha-1}}/\log n$. By Chebyshev, this occurs with probability at least

$$1 - c\frac{|T_m|}{\mu^m}n^{\frac{1}{\alpha-1}}\mu^{-m}(\log n)^2.$$

Connection probabilities: $\mathbb{P}_T\left(\mathbf{o} \overset{p_c}{\longleftrightarrow} T_n\right)$ - main term

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$$1 - c\frac{|T_m|}{\mu^m}n^{\frac{1}{\alpha-1}}\mu^{-m}(\log n)^2.$$

So need

$$cWn^{\frac{1}{\alpha-1}}\mu^{-m}(\log n)^2 \leq n^{-\varepsilon}.$$

Connection probabilities: $\mathbb{P}_T\left(\mathbf{o} \xleftrightarrow{p_c} T_n\right)$ - error term

Set $\boldsymbol{p}_{u,v} = \mathbb{P}_T\left(u \overset{*}{\leftrightarrow} T_n, v \overset{*}{\leftrightarrow} T_n\right)$. Bound p^{th} moment, $\frac{1}{2} < p < \frac{\alpha}{2}$:

$$\left(\sum_{\substack{u,v\in T_m\\ u\neq v}} \mathbb{P}_T\left(\mathbf{o} \leftrightarrow (u,v) \overset{*}{\leftrightarrow} T_n\right)\right)^p \leq \left(\sum_{i=1}^{m} \sum_{w\in T_i} \sum_{\substack{u,v\in T_m^{(w)}\\ u\neq v}} \frac{\boldsymbol{p}_{u,v}}{\mu^{2m-i}}\right)^p$$

Connection probabilities: $\mathbb{P}_T\left(\mathbf{o} \overset{p_c}{\longleftrightarrow} T_n\right)$ - error term

Set $\boldsymbol{p}_{u,v} = \mathbb{P}_T\left(u \overset{*}{\leftrightarrow} T_n, v \overset{*}{\leftrightarrow} T_n\right)$. Bound p^{th} moment, $\frac{1}{2} < p < \frac{\alpha}{2}$:

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Take expectation, apply Jensen's inequality many times and sum...

$$\mathbf{E}\left[\left(\sum_{\substack{u,v \in T_m \\ u \neq v}} \mathbb{P}_T\left(\mathbf{o} \leftrightarrow (u,v) \overset{*}{\leftrightarrow} T_n\right)\right)^p\right] \leq C\mu^{m(1-p)} \mathbf{E}\left[W^{2p}\right] \mathbf{E}[\boldsymbol{p}_{u,v}]^p$$

Connection probabilities: $\mathbb{P}_T\left(\mathbf{o} \xleftrightarrow{p_c} T_n\right)$ - error term

Set $\boldsymbol{p}_{u,v} = \mathbb{P}_T\left(u \xleftrightarrow{*} T_n, v \xleftrightarrow{*} T_n\right)$. Bound p^{th} moment, $\frac{1}{2} < p < \frac{\alpha}{2}$:

$$\left(\sum_{\substack{u,v\in T_m\\ u\neq v}} \mathbb{P}_T\left(\mathbf{o} \leftrightarrow (u,v) \xleftrightarrow{*} T_n\right)\right)^p \leq \left(\sum_{i=1}^{m} \sum_{w\in T_i} \sum_{\substack{u,v\in T_m^{(w)}\\ u\neq v}} \frac{\boldsymbol{p}_{u,v}}{\mu^{2m-i}}\right)^p$$

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$$\mathbf{E}\left[\left(\sum_{\substack{u,v\in T_m\\ u\neq v}} \mathbb{P}_T\left(\mathbf{o} \leftrightarrow (u,v) \xleftrightarrow{*} T_n\right)\right)^p\right] \leq C\mu^{m(1-p)}\mathbf{E}\left[W^{2p}\right]\mathbf{E}[\boldsymbol{p}_{u,v}]^p$$

$$\leq C'\mu^{m(1-p)} n^{-\frac{2p}{\alpha-1}}.$$

Connection probabilities: $\mathbb{P}_T\left(\mathbf{o} \overset{p_c}{\longleftrightarrow} T_n\right)$ - error term

Take expectation, apply Jensen's inequality many times and sum...

$$\mathbf{E}\left[\left(\sum_{\substack{u,v\in T_m\\ u\neq v}} \mathbb{P}_T\left(\mathbf{o} \leftrightarrow (u,v) \overset{*}{\leftrightarrow} T_n\right)\right)^p\right] \leq C' \mu^{m(1-p)} n^{-\frac{2p}{\alpha-1}}.$$

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Then L^p Markov inequality:

$$\mathbb{P}\left(\text{error} > n^{-\frac{1}{\alpha-1}} / \log n\right) \leq C'\mu^{m(1-p)} n^{-\frac{p}{\alpha-1}} (\log n)^p.$$

Connection probabilities: $\mathbb{P}_T\left(\mathbf{o} \overset{p_c}{\longleftrightarrow} T_n\right)$ - error term

Take expectation, apply Jensen's inequality many times and sum...

$$\mathbf{E}\left[\left(\sum_{\substack{u,v\in T_m\\ u\neq v}} \mathbb{P}_T\left(\mathbf{o} \leftrightarrow (u,v) \overset{*}{\leftrightarrow} T_n\right)\right)^p\right] \leq C' \mu^{m(1-p)} n^{-\frac{2p}{\alpha-1}}.$$

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So need $C' \mu^{m(1-p)} n^{-\frac{p}{\alpha-1}} (\log n)^p \leq n^{-\varepsilon}$.

Connection probabilities: $\mathbb{P}_T\left(\mathbf{o} \stackrel{p_c}{\longleftrightarrow} T_n\right)$

Need:

$$\begin{aligned} cWn^{\frac{1}{\alpha-1}}\mu^m(\log n)^2 &\leq n^{-\varepsilon} \\ \text{and } C'\mu^{m(1-p)}n^{-\frac{p}{\alpha-1}}(\log n)^p &\leq n^{-\varepsilon} \end{aligned}$$

Connection probabilities: $\mathbb{P}_T\left(\mathbf{o} \overset{p_c}{\longleftrightarrow} T_n\right)$

Need:

$$n^{\frac{1}{\alpha-1}}\mu^{-m} \ll n^{-\varepsilon}$$

$$\text{and } \mu^{m(1-p)}n^{-\frac{p}{\alpha-1}} \ll n^{-\varepsilon}$$

Connection probabilities: $\mathbb{P}_T\left(\mathbf{o} \overset{p_c}{\longleftrightarrow} T_n\right)$

Need:

$$n^{\frac{1}{\alpha-1}}\mu^{-m} \ll n^{-\varepsilon}$$
$$\text{and } \mu^{m(1-p)}n^{-\frac{p}{\alpha-1}} \ll n^{-\varepsilon}$$

Note that $1-p<p$ so choose m so that $\mu^m = n^{\frac{1+\varepsilon}{\alpha-1}}$ and $\mu^{m(1-p)/p} \leq n^{\frac{1-\varepsilon}{\alpha-1}}$.

And it's done!

Quenched vs annealed results - finite variance

$T_n = n^{th}$ generation of T, C = cluster of $\mathbf{o}$, $Y_n = |C \cap T_n|$,
$C_{\geq n} = C$ conditioned to have size n, $W = \lim \frac{Z_n}{m^n}$

Annealed

$$p_c = 1/\mu$$

$$\mathbb{P}\left(\mathbf{o} \overset{p_c}{\longleftrightarrow} T_n\right) \sim cn^{-1}$$

$$\mathbb{P}(|C| \geq n) \sim c' n^{-1/2}$$

Given $Y_n > 0$: $n^{-1} Y_n \overset{(d)}{\to} Y$

and $\left(n^{-1} Y_{n(1+t)}\right)_{t \geq 0} \overset{(d)}{\to} (Y_t)_{t \geq 0}$

Given $Y_\infty > 0$: $n^{-1} Y_n \overset{(d)}{\to} \hat{Y}$

$$(C_{\geq n}, n^{-1/2} d_n, \tfrac{1}{n}\mu_n) \underset{\text{GHP}}{\overset{(d)}{\to}} CRT$$

Quenched

$$p_c(T) = 1/\mu \text{ a.s.}$$

$$\mathbb{P}_T\left(\mathbf{o} \overset{p_c}{\longleftrightarrow} T_n\right) \sim W \cdot cn^{-1} \text{ a.s. } *$$

$$\mathbb{P}_T(|C| \geq n) \sim W \cdot c' n^{-1/2} \text{ a.s.}$$

$$n^{-1} Y_n^T \overset{(d)}{\to} Y \text{ a.s. } *$$

$$\left(n^{-1} Y_{n(1+t)}^T\right)_{t \geq 0} \overset{(d)}{\to} (Y_t)_{t \geq 0} \text{ a.s.}$$

$$n^{-1/2} Y_n^T \overset{(d)}{\to} \hat{Y} \text{ a.s. } *$$

$$(C_{\geq n}^T, n^{-1/2} d_n, \tfrac{1}{n}\mu_n) \underset{\text{GHP}}{\overset{(d)}{\to}} CRT \text{ a.s.}$$

*proved by Michelen (2019) under higher moment assumptions.

Extension to critical percolation on hyperbolic random planar maps??

C = cluster of $\mathbf{o}$, $C_{\geq n} = C$ conditioned to have size n,

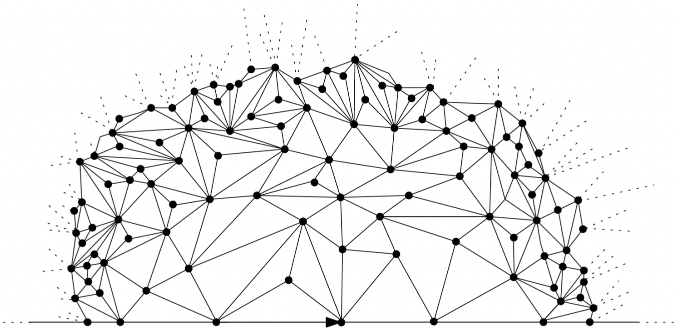


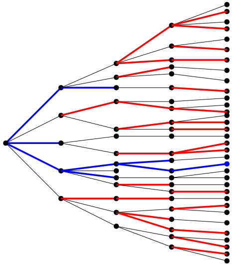
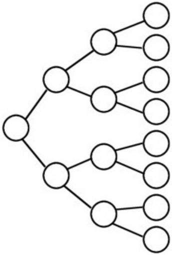
Image by Gourab Ray.

Extension to critical percolation on hyperbolic random planar maps??

C = cluster of **o**, $C_{\geq n} = C$ conditioned to have size n

Annealed	**Quenched**
p_c is explicit (Ray 2014)	
$\mathbb{P}(\text{Height}(C) \geq n) \sim cn^{-1}$ *	
$\mathbb{P}(\lvert C\rvert \geq n) \sim c' n^{-1/2}$ *	
$(C_{\geq n}, n^{-1/2} d_n, \frac{1}{n}\mu_n) \xrightarrow[\text{GHP}]{(d)} CRT$ *	

*A.-Croydon 2023.

Thank you!!