Fredrickson-Andersen 2 spin facilitated model

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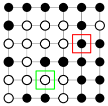
- the refresh occurs iff the site has at least 2 empty nearest neighbours $=$ *iff the kinetic constraint is satisfied*



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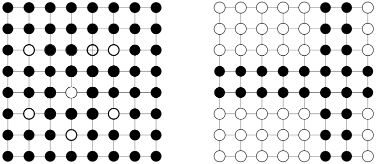

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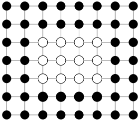

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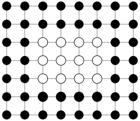

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Math. motivation #1: several IPS tools fail $\rightarrow$ new tools needed!

Motivations from physics

- FA2f was introduced in the '80's to model/understand the liquid/glass transition, a major open problem in physics
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- Math. motivation #2: put physicists works on firmer ground / settle controversies

Does $\eta \sim \mu_q$ contain a blocked cluster?

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How does the first time the origin is empty scale as $q \downarrow 0$?

$$\tau^{\text{BP}} = \exp\left(\lambda_d \, q^{-1/(d-1)}(1 + o(1))\right)$$ [Aizenman - Lebowitz '88]

- $\lambda_2 = \pi^2/18$ [Holroyd '08]
- $\lambda_d = \ldots \ \forall d > 2$ [Balogh Bollobas Duminil-Copin Morris '12]

Back to FA2f: our results

Theorem [Hartarsky, Martinelli, C.T. '20]

As $q \downarrow 0$, w.h.p. for the stationary FA-2f model on $\mathbb{Z}^d$ it holds

$$\tau = \exp\left(\frac{d \times \lambda_d}{q^{1/(d-1)}}(1 + o(1))\right), \quad d \geq 2$$

the same result holds for $\mathbb{E}_{\mu_q}(\tau)$. Thus, w.h.p. $\tau = (\tau^{\text{BP}})^{d+o(1)}$.

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Remark

- not a corollary of BP result: very different mechanism!
- we settle contrasting conjectures in physics literature

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Step 3 identify a bottleneck, i.e. an unlikely configuration set that has to be visited before emptying the origin $\Longrightarrow \tau \geq \ldots$

Heuristics ($q \downarrow 0$): optimal relaxation mechanism

- Relaxation is driven by the motion of rare large patches of empty sites, the droplets
- droplet density $\rho_D := \exp\left(-\frac{d\times\lambda_d}{q^{1/d-1}}(1+o(1))\right)$

  droplet length $L_D := 1/q^{\alpha}$, $\alpha > 2$
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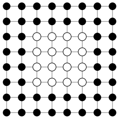

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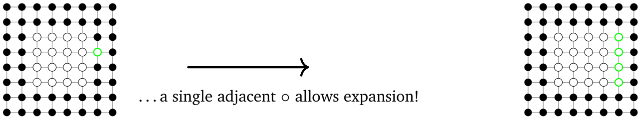

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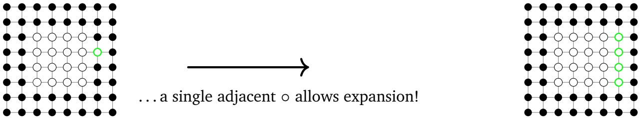

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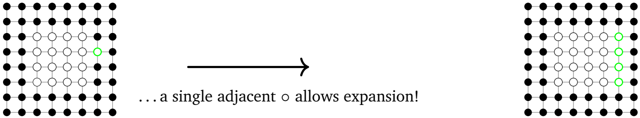

- Droplet motion requires few additional empty sites $\rightarrow$ this *good environment* is very likely since $L_D >> |\log q|/q$

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$\tau^{\text{BP}} \sim$ size of minimal region to be unblocked before the origin
= distance from the origin to the nearest droplet

$$\rightarrow \tau^{\text{BP}} \sim \rho_D^{-1/d} \sim \tau^{1/d}$$

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  - the supremum is over non constant functions;
  - $\mathcal{D}_{\text{FA-2f}}(f)$ is the "energy" $= \sum_\eta \mu(\eta) \sum_x c_x(\eta)(f(\eta^x) - f(\eta))^2$
  - η^x = configuration flipped at x; $c_x(\eta)$ = rate for $\eta \to \eta^x$

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- Renormalisation on droplet scale $\rightarrow$ CBSEP dynamics for the droplets and FA2f dynamics inside the droplets

$$T^{\mathrm{rel}}_{\text{FA-2f}} \le T^{\mathrm{rel}}_{\text{CBSEP}} \times T^{\mathrm{rel}}_{\text{FA-2f}}(L_D \,|\, \text{droplet})$$

- Prove Poincaré inequalities for FA2f in a droplet and CBSEP

$$T^{\mathrm{rel}}_{\text{CBSEP}} \le \rho_D^{-1}, \qquad T^{\mathrm{rel}}_{\text{FA-2f}}(L_D \,|\, \text{droplet}) \le e^{\frac{|\log q|^3}{\sqrt{q}}} \ll T^{\mathrm{rel}}_{\text{CBSEP}}$$

Why bothering with the exact constants?

- sharp constant is extremely hard to grasp numerically:
  - subtle corrections to the dominant behavior, ex. $d = 2$: $\tau^{\text{BP}} = \exp\left(\frac{\pi^2}{18q}(1 - c\sqrt{q})\right)$ (Hartarsky, Morris) $\rightarrow$ slow convergence
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- a deeper understanding of the cooperative relaxation
- the mathematical tools we build are very flexible
  $\rightarrow$ we adapt them to get universality results for KCM in $d = 2$

Open problems

- Conjecture $\#1$: Start from $\mu_{q'}$, with $q' \neq q$ and $q, q' > 0$ and call μ^t the evoluted measure at time t, $\mu^t = \mu_{q'} P_t$. It holds

$$\lim_{t\to\infty} \mu^t = \mu_q.$$

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- Conjecture #2: Consider FA-2f on $\mathbb{Z}_d^+$ with empty b.c. and start from a completely filled configuration. The set of sites that have been already updated at time t rescaled by t converges as $t \to \infty$ to a non random limit shape.

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More generally: we lack robust tools to tackle the out of equilibrium regime of KCM !

Thanks for your attention!

Droplets

A multi-scale definition

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  - R is a single site for $n = 0$;
  - $R = \ell_m \times h$ with $h \in (\ell_{m-1}, \ell_m]$ for $n = 2m$;
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- Super-good (SG) rectangles:
  - a rectangle of class 0 is SG if it is empty;
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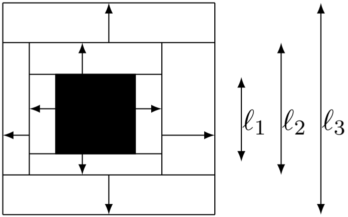
Droplets are defined as $\ell_N \times \ell_N$ SG rectangles

How do droplets look like?

Ex. of a SG rectangle of class 6. Here arrows indicate traversability and the black square is completely empty.



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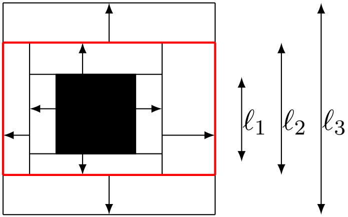

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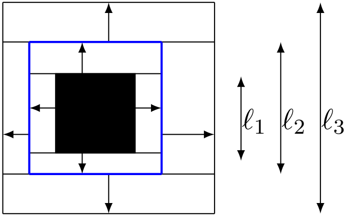

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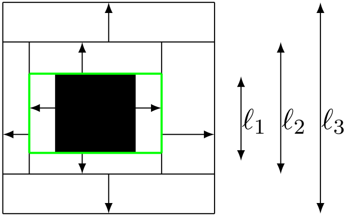

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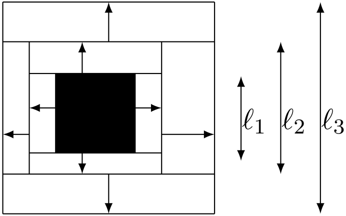

Droplets $= \ell_N \times \ell_N$ squares that are SG with $N = 8|\log q|/\sqrt{q}$, so that $\ell_N = q^{-17/2+o(1)}$

NB any other big enough power would work the same, no special meaning of $17/2\ldots$

FA-jf model

- for jn-BP for all $d \geq j \geq 2$, w.h.p. it holds

$$\tau_0^{\text{BP}} \sim \exp^{j-1}\left(\frac{\tilde{\lambda}_{d,j}}{q^{1/(d-j+1)}}\right)$$

$\exp^k$ = exponential iterated k times (Balogh, Bollobas, Duminil-Copin, Morris '12)

Same scaling for τ_0 (Hartarsky, Martinelli, C.T. in progress)

- $j = 1$: $\tau_0^{\text{BP}} = 1/q^{1/d}$, $\tau_0 = 1/q^{\nu(d)}$, $\nu(1) = 3$, $\nu(d) = 2$ $d \geq 2$ (log corrections in $d = 2$) (Cancrini, Roberto, Martinelli, C.T. '08 + Shapira '20)

- $d < j$: $\tau_0 = \tau_0^{\text{BP}} = \infty$ w.h.p. for $q \to 0$

What happens if we change constraint?

Universality results for $d = 2$

1. Supercritical unrooted: $\tau(q) = q^{-\Theta(1)}$
2. Supercritical rooted:
   $\tau(q) = q^{-\Theta(1)|\log q|}$
3. Finitely critical: $\tau(q) = \exp\left(\frac{\Theta(1)(\log q)^{\Theta(1)}}{q^{\nu}}\right)$
4. Infinitely critical:
   $\tau(q) = \exp\left(q^{-2\nu}(\log q)^{c}\right)$
5. Subcritical:
   $\exists\, q_c > 0$, s.t. for $q < q_c$ it holds $\tau(q) = \infty$

[I.Hartarsky, L.Marêché, F.Martinelli, R. Morris, C.T.]

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[I.Hartarsky, L.Marêché, F.Martinelli, R. Morris, C.T.]

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  logarithmic energy barrier: droplet are at distance $\ell \sim \tau^{\mathrm{BP}}$ from the origin and must create $\sim \log \ell$ droplets to reach it [Marêché, Martinelli, C.T. '20]

The g-CBSEP chain

- $G = (V, E)$: finite connected graph
- $(\mathcal{S}, \pi)$: finite probability space
- $\mathcal{S} = \mathcal{S}_0 \sqcup \mathcal{S}_1$ and $\rho = \pi(\mathcal{S}_1)$
- given $\sigma \in \mathcal{S}^V$, $x \in V$ is *occupied* iff $\sigma_x \in \mathcal{S}_1$
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Theorem [Hartarsky, Martinelli, C.T. '20]

As $\rho \downarrow 0$, $T_{\text{rel}}^{\text{g-CBSEP}} \leq O(\rho^{-1} \log(1/\rho))$