Long and pointy Poisson-Voronoi cells

Distribution tail of the circumradius of the typical cell

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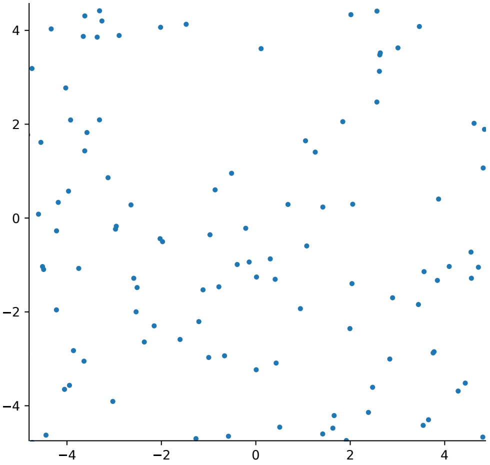
(0) Introduction and setup of the problem

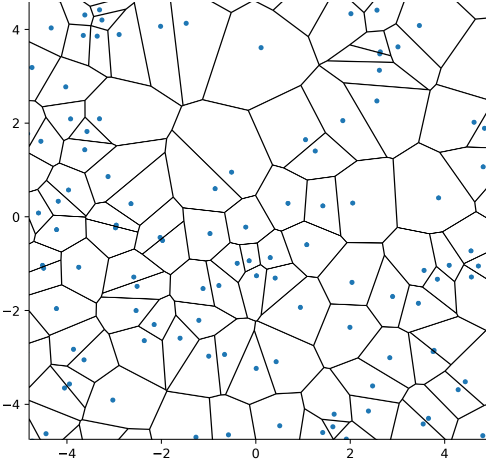
Construction of the homogeneous Poisson-Voronoi tessellation 1/2

Φ a Poisson point process on $\mathbb{R}^d$ with intensity measure the Lebesgue measure.



Construction of the homogeneous Poisson-Voronoi tessellation 2/2

To $X \in \Phi$ associate the cell $Cell(X, \Phi) = \{y \in \mathbb{R}^d : \forall X' \in \Phi, \|y - X\| \leq \|y - X'\|\}$.



The collection of all cells constitutes the homogeneous Poisson-Voronoi tessellation.

The typical cell and some observables

Typical cell

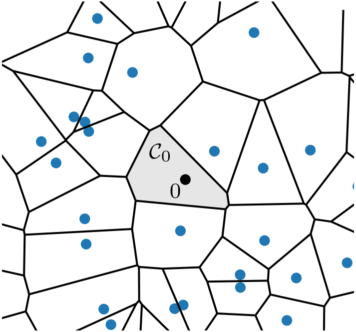
$$\mathcal{C}_0 = \mathcal{C}(0, \Phi \cup \{0\}),$$

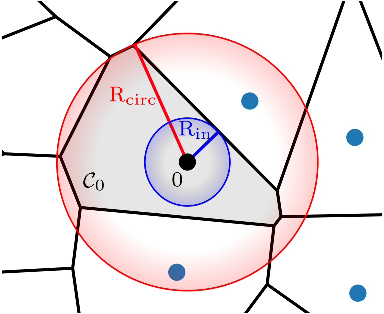
circumradius of the typical cell

$$R_{\text{circ}} = \min\{r \geq 0 : \mathcal{C}_0 \subseteq \mathcal{B}_r(0)\},$$

inradius of the typical cell

$$R_{\text{in}} = \max\{r \geq 0 : \mathcal{B}_r(0) \subseteq \mathcal{C}_0\}.$$

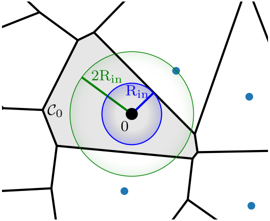




Previous works

Explicit law of the inradius:

$$\mathbb{P}\left(R_{\text{in}} \geq t\right) = \mathbb{P}(\Phi \cap \mathcal{B}(0, 2t) = \emptyset) = e^{-\text{Vol}_d(\mathcal{B}(0,2t))} = e^{-\kappa_d (2t)^d}.$$



In [1], double bound on the tail probability of the circumradius. For the 2-dimensional case, for t large enough,

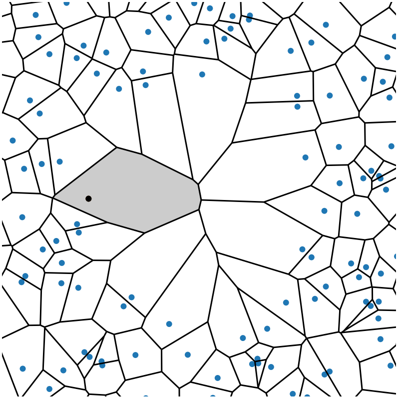
$$2\pi t^2 e^{-\pi t^2} \leq \mathbb{P}\left(R_{\text{circ}} \geq t\right) \leq 4\pi t^2 e^{-\pi t^2}.$$

Our question:
$\mathbb{P}(R_{\text{circ}} \geq t) \overset{t\to\infty}{\sim}$?

[1] P. Calka. The distributions of the smallest disks containing the Poisson-Voronoi typical cell and the Crofton cell in the plane. *Adv. in Appl. Probab.* **34**, 702-717 (2002)

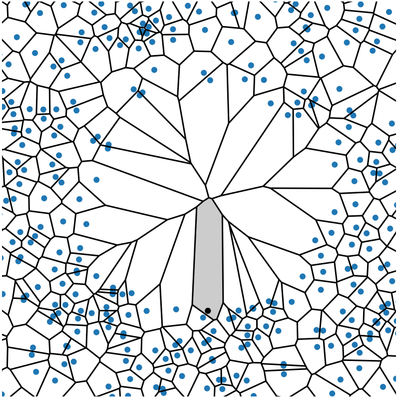
Simulations of a cells with large circumradius 1/3

A cell with circumradius ≥ 3:



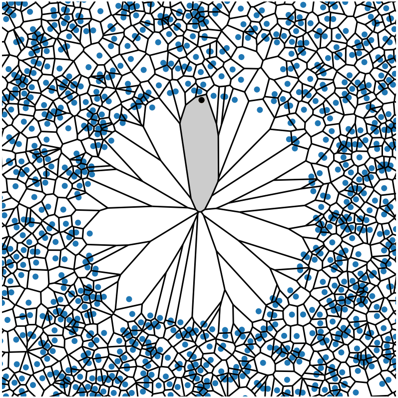
Simulations of a cells with large circumradius 2/3

A cell with circumradius ≥ 5:



Simulations of a cells with large circumradius 3/3

A cell with circumradius ≥ 10:



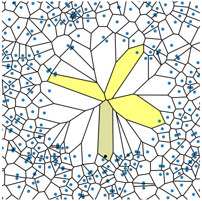
Guess and implications

Our guess:

$$\mathbb{P}(R_{\text{circ}} \geq t) \underset{t\to\infty}{\sim} C_d\, t^{n_d}\, e^{-\kappa_d t^d}$$

with C_d a constant, n_d a polynomial in d, $\kappa_d = \text{Vol}_d(\mathcal{B}_{\mathbb{R}^d})$.

A cluster of cells with circumradius ≥ 5:



$$\text{extremal index} = \frac{1}{\mathbb{E}\left[\#\text{ exceedances in a cluster of exceedances }\right]} \underset{[2]}{\overset{\text{conj.}}{=}} \frac{1}{2d}$$

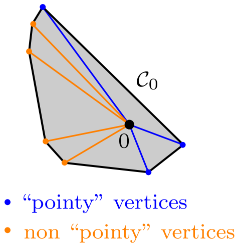
The extremal index for the circumradius is hidden in C_d.

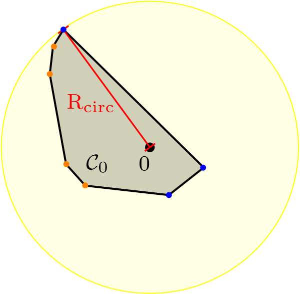
[2] P. Calka & N. Chenavier. Extreme values for characteristic radii of a Poisson-Voronoi tessellation. *Extremes* **17**, 359-385 (2014)

The circumradius as the distance to a "pointy" vertex

Notation. A vertex of $\mathcal{C}_0$ is said to be "pointy" if, locally around it, it is the point of $\mathcal{C}_0$ maximizing the distance to the nucleus 0.

$$R_{\text{circ}} = \text{dist}\,(0,\ \text{one of the "pointy" vertices of } \mathcal{C}_0)$$





Idea

Consider

$$\mathcal{V}_{\max}^{\geq t} = \{ \text{ all "pointy" vertices of } \mathcal{C}_0 \text{ at distance } \geq t \text{ from } 0 \text{ } \}$$

then

$$\{R_{\text{circ}} \geq t\} = \{\#\mathcal{V}_{\max}^{\geq t} \geq 1\}.$$

We access the tail of the distribution of R_{circ} from the study of $\#\mathcal{V}_{\max}^{\geq t}$:

$$\mathbb{P}(R_{\text{circ}} \geq t) = \mathbb{P}(\#\mathcal{V}_{\max}^{\geq t} \geq 1) \underbrace{\overset{t\to\infty}{\sim}}_{(**)} \underbrace{\mathbb{E}\left[\#(\mathcal{V}_{\max}^{\geq t})\right]}_{(*)}.$$

(*) is computed in sections (1),(2).
(**) arises from the asymptotic negligibility of $\mathbb{E}[\#((\mathcal{V}_{\max}^{\geq t})^2_{\neq})]$ with respect to $\mathbb{E}[\#(\mathcal{V}_{\max}^{\geq t})]$ as $t \to \infty$: section (3).

Outline

To show:

$$\mathbb{P}\left(R_{\mathrm{circ}} \geq t\right) \underset{t\to\infty}{\sim} \mathbb{E}\left[\#(\mathcal{V}_{\max}^{\geq t})\right] = C_d t^{n_d} e^{\kappa_d t^d}.$$

(0) Introduction and setup

(1) Computation of $\mathbb{E}[\#(\mathcal{V}_{\max}^{\geq t})]$ up to multiplication by a constant: find n_d

(2) Computation of the expected volume of a random simplex: find C_d

(3) Negligibility of $\mathbb{E}[\#((\mathcal{V}_{\max}^{\geq t})^2_{\neq})]$ and Theorem

(1) Setup of $\mathbb{E}[\#(\mathcal{V}_{\max}^{\geq t})]$: finding n_d

Characterization of "pointy" vertices

Lemma 1. Let c be a vertex of $\mathcal{C}_0$ at equal distance from 0 and $X_1, ..., X_d \in \Phi$ and

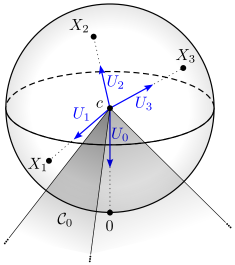
$$U_0 = \frac{c}{\|c\|} \text{ and } U_i = \frac{X_i - c}{\|X_i - c\|} \text{ for } i = 1..d$$

then, c is "pointy" iff

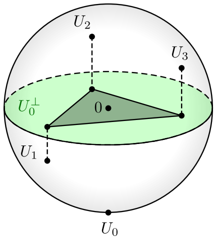
$$0 \in \text{Conv}\left(P_{U_0^\perp}(U_1), .., P_{U_0^\perp}(U_d)\right)$$

where $P_{U_0^\perp}$ denotes the orthogonal projection onto $U_0^\perp$ and Conv the convex hull of the specified points.

"pointy" vertex c



$0 \in \text{Conv}(P_{U_0^\perp}(U_1), P_{U_0^\perp}(U_2), P_{U_0^\perp}(U_3))$



The Mecke formula

For any positive function f,

$$\mathbb{E}\left[\sum_{(X_1,..,X_d)\in\Phi^d_{\neq}} f(X_1,...,X_d,\Phi)\right] = \int_{(\mathbb{R}^d)^d} dx_1..dx_d\,\mathbb{E}\left[f(x_1,...,x_d,\Phi\cup\{x_1,..,x_d\})\right]$$

where $\Phi^d_{\neq}$ is the set of d-tuples of pairwise distinct points of Φ.

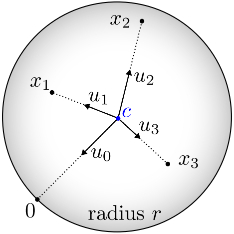
Blaschke-Petkantschin spherical change of variables

For any positive function f of d points of $\mathbb{R}^d$,

$$\int_{(\mathbb{R}^d)^d} dx_1..dx_d\, f(x_1,...,x_d)$$

$$= \int_0^\infty dr \int_{(\mathcal{S}_{\mathbb{R}^d})^{d+1}} du_0..du_d\, d!\, r^{d^2-1}\Delta_d(u_0,..,u_d) f(r(-u_0+u_1),..,r(-u_0+u_d))$$

where $\Delta_d(u_0,..,u_d) = \mathrm{Vol}_d(\mathrm{Conv}(u_0,..,u_d))$.



Setup of the expectation of the number of pointy vertices at large distance

$$\mathbb{E}\left[\#\mathcal{V}_{\max}^{\geq t}\right] = \frac{1}{d!}\mathbb{E}\left[\sum_{(X_1,..,X_d)\in\Phi^d_{\neq}} 1_{\text{Center}(\text{Ball}(0,X_1,..,X_d))\in\mathcal{V}_{\max}^{\geq t}}\right]$$

$$\overset{\text{Mecke}}{=} \frac{1}{d!}\int_{(\mathbb{R}^d)^d} dx_1...dx_d\mathbb{E}\left[1_{\underbrace{\text{Center}(\text{Ball}(0,x_1,..,x_d))}_{=c}\in\mathcal{V}_{\max}^{\geq t}(\Phi\cup\{x_1,..,x_d\})}\right]$$

$$\overset{\text{B.-P.}}{=} \frac{1}{d!}\int_0^\infty dr \int_{(\mathcal{S}_{\mathbb{R}^d})^{d+1}} du_0...du_d d!\, r^{d^2-1}\Delta_d(u_0,..,u_d)\mathbb{E}\left[1_{c\in\mathcal{V}_{\max}^{\geq t}(\Phi\cup\{x_1,..,x_d\})}\right]$$

$$\overset{\text{Fubini}}{=} \int_0^\infty dr r^{d^2-1}e^{-\kappa_d r^d}1_{r\geq t}$$

$$\int_{(\mathcal{S}_{\mathbb{R}^d})^{d+1}} du_0...du_d\Delta_d(u_0,..,u_d)1_{0\in\text{Conv}(P_{u_0^\perp}(u_1),..,P_{u_0^\perp}(u_d))}$$

Summary and value of n_d

Denoting $\kappa_d = \mathrm{Vol}_d(\mathcal{B}_{\mathbb{R}^d})$, $\sigma_d = d\kappa_d = \mathrm{Vol}_{d-1}(\mathcal{S}_{\mathbb{R}^d})$ and $U_0, .., U_d$ i.i.d. $\mathrm{Unif}(\mathcal{S}_{\mathbb{R}^d})$,

$$\mathbb{E}\left[\#(\mathcal{V}_{\max}^{\geq t})\right] = \left(\frac{1}{d\kappa_d}\sum_{i=0}^{d-1} t^{di}\kappa_d^{i-d+1}\frac{(d-1)!}{i!}e^{-\kappa_d t^d}\right)\sigma_d^{d+1}\mathbb{E}\left[\Delta_d(U_0, .., U_d)1_{0\in\mathrm{Conv}(P_{U_0^\perp}(U_1,..,U_d))}\right]$$

$$\underset{t\to\infty}{\sim} C_d t^{d(d-1)} e^{-\kappa_d t^d}$$

so

$$n_d = d(d-1),$$

$$C_d = \sigma_d^d\mathbb{E}\left[\Delta_d(U_0, .., U_d)1_{0\in\mathrm{Conv}(P_{U_0^\perp}(U_1,..,U_d))}\right].$$

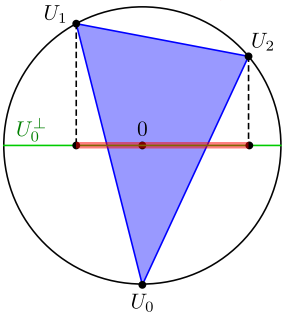
(2) Expectation of a random simplex: finding C_d

Previous work (Miles)

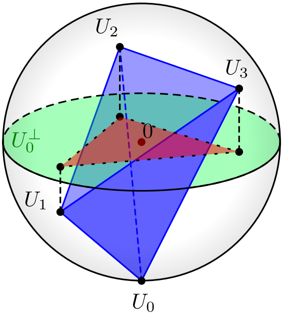
Our question:

$$\mathbb{E}\left[\Delta_d(U_0,..,U_d)1_{0\in \mathrm{Conv}(P_{U_0^\perp}(U_1,..,U_d))}\right] = ?$$

$\Delta_2(U_0,U_1,U_2)1_{0\in \mathrm{Conv}(P_{U_0^\perp}(U_1,U_2))}$



$\Delta_3(U_0,U_1,U_2,U_3)1_{0\in \mathrm{Conv}(P_{U_0^\perp}(U_1,U_2,U_3))}$



By [3], for $U_0,..,U_d$ i.i.d. uniformly distributed random variables on the $(d-1)$-dimensional sphere $\mathcal{S}_{\mathbb{R}^d}$:

$$\mathbb{E}\left[\Delta_d(U_0,..,U_d)\right] = \frac{1}{\sqrt{\pi}d!}\frac{\Gamma(\frac{d}{2})\Gamma(\frac{d^2+1}{2})}{\Gamma(\frac{d^2}{2})}\left(\frac{\Gamma(\frac{d}{2})}{\Gamma(\frac{d+1}{2})}\right)^d$$

[3] R.E. Miles. Isotropic random simplices. *Adv. in App. Prob.* **Vol. 3, No. 2**, 353-382 (1971)

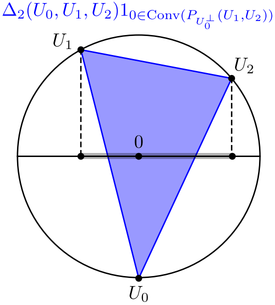
Reduction to the expectation of a subsimplex

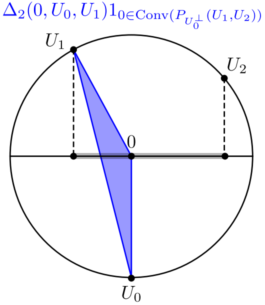
From now on, we use the short notations

$$\mathbf{a}_{1..k} = (a_1, .., a_k) \text{ and } f(\mathbf{a}_{1..k}) = (f(a_1), .., f(a_k)).$$

Lemma 2. For $\mathbf{U}_{0..d}$ i.i.d. Unif$(\mathcal{S}_{\mathbb{R}^d})$:

$$\mathbb{E}\left[\Delta_d(\mathbf{U}_{0..d})1_{0\in \text{Conv}(P_{U_0^\perp}(\mathbf{U}_{1..d}))}\right] = d\mathbb{E}\left[\Delta_d(0, \mathbf{U}_{0..d-1})1_{0\in \text{Conv}(P_{U_0^\perp}(\mathbf{U}_{1..d}))}\right]$$





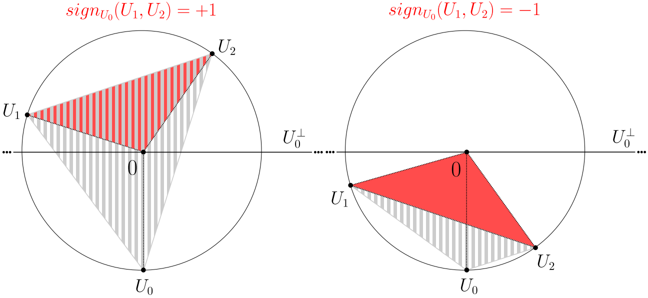
Sketch of proof for Lemma 2., d=2

For U_0, U_1, U_2 i.i.d. Unif$(\mathcal{S}_{\mathbb{R}^2})$, under the pointy condition

$$(*) = 0 \in \mathrm{Conv}(P_{U_0^\perp}(U_1), P_{U_0^\perp}(U_2))$$

we can write the signed sum

$$\Delta_2(U_0,U_1,U_2)1_{(*)} = \underbrace{\Delta_2(0,U_0,U_1)1_{(*)} + \Delta_2(0,U_0,U_2)1_{(*)}}_{\mathbb{E}[\cdot]=2\mathbb{E}[\Delta_2(0,U_0,U_1)1_{(*)}]} + \underbrace{\Delta_2(0,U_1,U_2)sign_{U_0}(U_1,U_2)1_{(*)}}_{\mathbb{E}[\cdot]=0}.$$

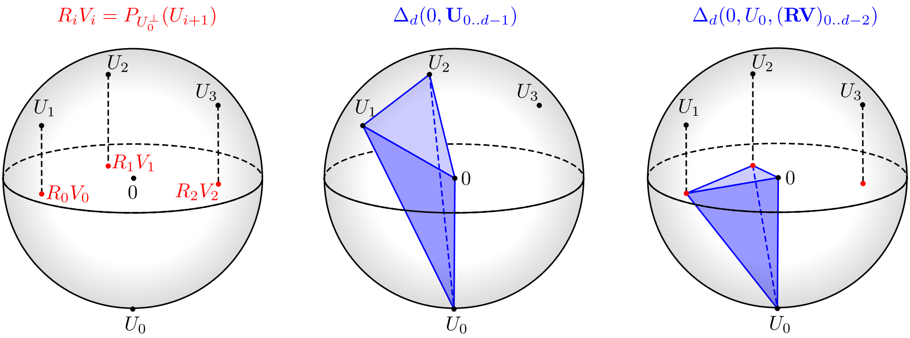


Recursive relation on the expectation of the subsimplex

Lemma 3. Let $\mathbf{U}_{0..d}$ be i.i.d. Unif($\mathcal{S}_{\mathbb{R}^d}$) and $\mathbf{V}_{0..d-1}$ be i.i.d. Unif($\mathcal{S}_{\mathbb{R}^{d-1}}$). Then

$$\mathbb{E}\left[\Delta_d(0,\mathbf{U}_{0..d-1})1_{0\in\mathrm{Conv}(P_{\mathrm{U}_0^\perp}(\mathbf{U}_{1..d}))}\right] = \frac{1}{2d}\left(\frac{B(\frac{d}{2},\frac{1}{2})}{B\left(\frac{d-1}{2},\frac{1}{2}\right)}\right)^{d-1}\mathbb{E}\left[\Delta_{d-1}(0,\mathbf{V}_{0..d-2})1_{0\in\mathrm{Conv}(P_{\mathrm{V}_0^\perp}(\mathbf{V}_{1..d-1}))}\right]$$

where $B(\cdot,\cdot)$ denotes the Beta function.



The expectation of the number of "pointy" vertices at large distance

Proposition 1. Let $\mathbf{U}_{0..d}$ be i.i.d. uniform random variables on the $(d-1)$-dimensional sphere $\mathcal{S}_{\mathbb{R}^d}$, then

$$\mathbb{E}\left[\Delta_d(\mathbf{U}_{0..d})1_{0\in \mathrm{Conv}(P_{\mathrm{U}_0^\perp}(\mathbf{U}_{1..d}))}\right] = \frac{1}{\sqrt{\pi}2^{d-1}(d-1)!}\frac{\Gamma\left(\frac{d}{2}\right)^d}{\Gamma\left(\frac{d+1}{2}\right)^{d-1}}$$

and therefore,

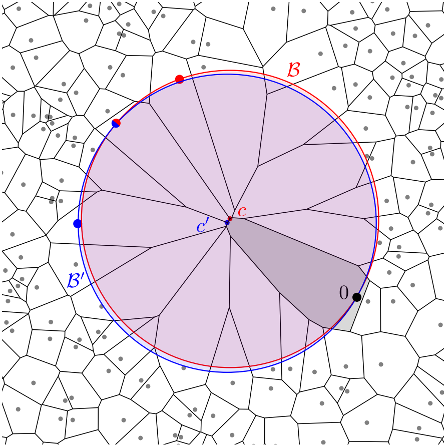
$$\mathbb{E}\left[\#(\mathcal{V}_{\max}^{\geq t})\right] \underset{t\to\infty}{\sim} C_d t^{d(d-1)} e^{-\kappa_d t^d}$$

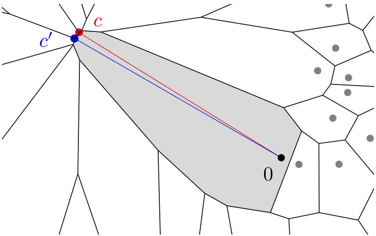
where

$$C_d = \frac{2\pi^{\frac{d^2-1}{2}}}{(d-1)!\Gamma\left(\frac{d+1}{2}\right)^{d-1}}$$

(3) Negligibility of $\mathbb{E}[\#((\mathcal{V}_{\max}^{\geq t})_{\neq}^2)]$ and Theorem

Most likely realization of a distinct pair of "pointy" vertices





Negligibility of $\mathbb{E}\left[\#\left(\mathcal{V}_{\max}^{\geq t}\right)_{\neq}^2\right]$ and Theorem

Proposition 2. As $t \to \infty$,

$$\mathbb{E}\left[\#\left(\mathcal{V}_{\max}^{\geq t}\right)_{\neq}^{2}\right] = \mathcal{O}\left(t^{d(d-2)}e^{-\kappa_d t^d}\right) = o\left(\mathbb{E}\left[\#\mathcal{V}_{\max}^{\geq t}\right]\right).$$

Theorem. As $t \to \infty$,

$$\mathbb{P}(R_{\text{circ}} \geq t) = C_d t^{d(d-1)} e^{-\kappa_d t^d} + \mathcal{O}(t^{d(d-2)} e^{-\kappa_d t^d})$$

where $\kappa_d = \frac{\pi^{\frac{d}{2}}}{\Gamma(\frac{d}{2}+1)}$ is the volume of the d-dimensional unit ball and

$$C_d = \frac{2\pi^{\frac{d^2-1}{2}}}{(d-1)!\Gamma(\frac{d+1}{2})^{d-1}}$$

Thank you for your attention!