Weierstrass Institute for
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Cluster sizes in subcritical soft Boolean models

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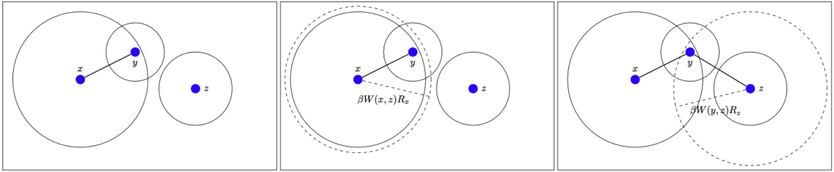
- *Vertex locations:* X homogeneous Poisson point process on $\mathbb{R}^d$ with intensity 1
- *Vertex marks:* R_x iid for $x \in X$ with $\mathbb{P}(R_x > r) = 1 \wedge r^{-d/\gamma}$ and $0 < \gamma < 1$
- *Edge weights:* $W_{x,y}$ iid for $x, y \in X$ with $\mathbb{P}(W_{x,y} > r) = 1 \wedge r^{-d\delta}$ and $\delta > 1$
- *Edge drawing:* $x, y \in X$ connected by edge iif

$$|x-y| \le \beta W(x,y)\big(R_x \vee R_y\big), \qquad \beta > 0$$



- *Boolean model* subgraph: $W_{x,y} \equiv 1$ and $R_x \vee R_y \le R_x + R_y \le 2(R_x \vee R_y)$
- *Random connection model* subgraph: $R_x \equiv 1$
- *Connected component* of origin (o, R_o)

$$\mathscr{C}_\beta := \{x \in X : o \leftrightarrow x \text{ in } \mathscr{G}_\beta\}$$

- *Critical percolation threshold:*

$$\beta_c := \beta_c(\gamma, \delta) := \sup\{\beta > 0 : \mathbb{P}_o(\sharp\mathscr{C}_\beta = \infty) = 0\},$$

Nontriviality: $\beta_c > 0$ if $\gamma < \delta/(\delta+1)$; $\beta_c = 0$ if $\gamma > \delta/(\delta+1)$

Gracar, Lüchtrath, Mörters: Percolation phase transition in WDRCMs, AdvAP 2021

- *Cluster diameter and cardinality:*

$$\mathscr{D}_\beta := \sup\{|x| : x \in \mathscr{C}_\beta\} \quad \textit{and} \quad \mathscr{N}_\beta := \sharp\mathscr{C}_\beta$$

Theorem (Subcritical diameter)

Let $d \geq 1, \delta > 1$, and $0 < \gamma < \delta/(\delta+1)$. Then, there exists $\beta_0 := \beta_0(\gamma, \delta) > 0$ such that, for all $0 < \beta < \beta_0$, there exist constants $0 < c, C < \infty$, depending on β, γ, and δ, such that for all $m > 1$,

(i) *if $\gamma < 1/(\delta+1)$, we have*

$$cm^{d(1-\delta)} \leq \mathbb{P}_o(\mathscr{D}_\beta > m) \leq Cm^{d(1-\delta)},$$

(ii) *if $\gamma = 1/(\delta+1)$, we have*

$$cm^{d(1-\delta)} \leq \mathbb{P}_o(\mathscr{D}_\beta > m) \leq C\log(m)m^{d(1-\delta)}, \text{ and}$$

(iii) *if $1/(\delta+1) < \gamma < \delta/(\delta+1)$, we have*

$$cm^{d\left(1-\frac{\delta-1+\gamma}{\gamma\delta}\right)} \leq \mathbb{P}_o(\mathscr{D}_\beta > m) \leq C\log(m)^{1\vee d(\delta-1)}m^{d\left(1-\frac{\delta-1+\gamma}{\gamma\delta}\right)}.$$

Comments

- $\gamma \downarrow 0$ recovers *long-range percolation* exponent $d(1-\delta)$
- $\delta \uparrow \infty$ recovers *classical Boolean model* exponent $d(1-1/\gamma)$
  Gouéré: Subcritical regimes in the Poisson Boolean model of continuum percolation. AP (2008)
- Case (i) shows *long-range* dominates
- Case (iii) shows *mixed behavior*
- We believe that $\beta_0 = \beta_c$ and have some *bounds on* β_0 based on system parameters
- Phase transition of *2nd order* in γ
- Case (i) and (iii) hold under *addition of slowly varying functions* to the Paretos, but then also slowly varying correction terms in the bounds
- *Logarithmic correction* in upper bound of Case (ii) and (iii) should be absent
- *Lower bounds true* for all β

- *Qualitative difference* due to long-range effect: Single long edge can increase diameter but not cardinality; cardinality driven by high-degree nodes

Theorem (Subcritical cardinality)

Let $d \geq 1$, $\delta > 1$, and $0 < \gamma < 1$.

(i) *If $\gamma < 1/2$, then there exists $\beta_0 := \beta_0(\gamma, \delta) > 0$ such that, for all $0 < \beta < \beta_0$, there exist constants $0 < c, C < \infty$ such that, for all $m > 1$,*

$$cm^{1-1/\gamma} \leq \mathbb{P}_o(\mathscr{N}_\beta > m) \leq Cm^{1-1/\gamma}.$$

(ii) *If $\gamma > 1/2$, then $\mathbb{E}_o \mathscr{N}_\beta = \infty$ for all $\beta > 0$.*

- *Qualitative difference* due to long-range effect: Single long edge can increase diameter but not cardinality; cardinality driven by high-degree nodes

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(ii) *If $\gamma > 1/2$, then $\mathbb{E}_o\mathscr{N}_\beta = \infty$ for all $\beta > 0$.*

- *Same exponent* as in classical Boolean model (volume = cardinality)
- *Different order* as in long-range percolation: exponentially-tailed degrees
- Possible that $\mathbb{E}_o\mathscr{N}_\beta < \infty$ and $\mathbb{E}_o\mathscr{D}^d_\beta = \infty$ but $\mathbb{E}_o\mathscr{D}^d_\beta < \infty$ implies $\mathbb{E}_o\mathscr{N}_\beta < \infty$
- *Lower bound true* for all β
- Precise decay in *finite-variance regime* but *should hold* for all $\gamma < \delta/(\delta + 1)$
- Result true for *scale-free percolation* ($R_x \vee R_y \to R_x R_y$; subcritical phase $\gamma \leq 1/2$)

- Lower bound in Theorem I:
  - Part (i): Diameter driven by *longest edge incident to origin* in longe-range percolation (Boolean part plays no role)
  - Part (iii): Diameter driven by *longest long-range edge of powerful Boolean neighbor* of origin (mixture of Boolean and long-range contribution)
- Upper bound in Theorem I:
  - Part (i): *No better strategy* available as used in lower bound
  - Part (iii): Build *skeleton path* of important vertices and *first-moment method*
- Theorem II:
  - Coupling with scale-free percolation and multitype branching process (with finite second moment)

- *Case (i) and (ii):* Number of long-range neighbors $\geq m$ is Poisson distributed,

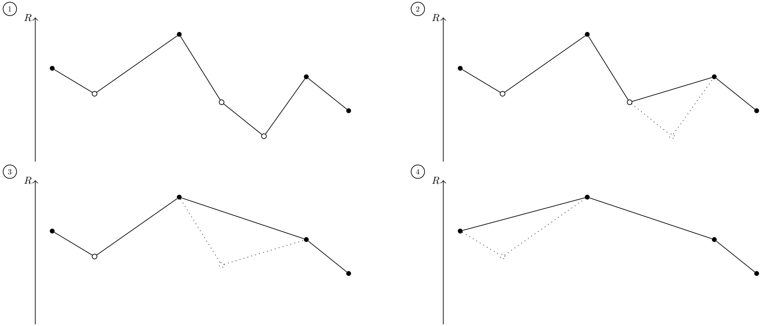
$$\mathbb{P}_o(\mathscr{D}_\beta > m) \geq \mathbb{P}_o(\exists x \sim_{\text{long-range}} o \colon |x| > m) \geq 1 - \exp(-cm^{d(1-\delta)})$$

- *Case (iii):* Search for Boolean nearest neighbor with weight $R_x \geq m^{(\delta-1)/\delta}$, which has a long-range neighbor $\geq m$ with typical mark, i.e.,

$$\begin{aligned}\mathbb{P}_o(\mathscr{D}_\beta > m) &\geq \mathbb{P}_o(\exists x \sim o, y \sim x \colon |x| < m,\ R_x \geq m^{(\delta-1)/\delta},\ |y| > m)\\ &\geq 1 - \exp(-cm^{d(1-(\delta-1+\gamma)/(\gamma\delta))})\end{aligned}$$

- If $\gamma > 1/2$, *expected number of length-2 paths* $= \infty$ since 2nd moment is absent
- Introduce *skeleton paths* of powerful vertices (paths of running maxima from two sides)

  Gracar, Lüchtrath, Mörters: Percolation phase transition in WDRCMs, AdvAP 2021



- Probability of connection of *two consecutive skeleton vertices* via connector path bounded by constant times direct connection probability

- *First-moment bound* only on skeletons
- If *γ<1/(δ+1)*: γ manifests in constant only
- If *γ>1/(δ+1)*: Skeleton approach allow to focus on skeleton paths

A Existence of *powerful vertex in origin's component* has right order

$$\mathbb{P}_o(\exists x \in \mathscr{C}_\beta : R_x \geq m^{(\delta-1)/\delta}) \leq c m^{d(1-(\delta-1+\gamma)/(\gamma\delta))}$$

B Existence of *two-step path with weak vertices* has right order

$$\mathbb{P}_o(\exists x \in \mathscr{C}_\beta, y \sim x \colon |x| \leq m,\ |y| \geq 2m,\ R_x, R_y < m^{(\delta-1)/\delta}) \\ \leq c m^{d(1-(\delta-1+\gamma)/(\gamma\delta))}$$

C Existence of *two-step path with distant weak vertices* has (almost) right order

$$\mathbb{P}_o(\exists x \in \mathscr{C}_\beta, y \sim x \colon |x|, |y| \leq 2m,\ |x-y| > m/\log(m),\ R_x, R_y < m^{(\delta-1)/\delta}) \\ \leq c \log(m)^{d(\delta-1)} m^{d(1-(\delta-1+\gamma)/(\gamma\delta))}$$

- $\mathbb{P}_o(\mathscr{D}_\beta > m) \leq \mathbb{P}_o(A) + \mathbb{P}_o(\{\mathscr{D}_\beta > m\} \cap A^c)$
- With $D = \{\exists x \in \mathscr{C}_\beta : m < |x| < 2m,\ R_x < m^{(\delta-1)/\delta}\}$,

$$\begin{aligned}\mathbb{P}_o(\{\mathscr{D}_\beta > m\} \cap A^c) &\leq \mathbb{P}_o(D) + \mathbb{P}_o(B) \\ &\leq \mathbb{P}_o(D \cap C^c) + \mathbb{P}_o(C) + \mathbb{P}_o(B)\end{aligned}$$

- $D \cap C^c$ implies existences of (weak vertex) path of length $\log m$ and hence

$$\mathbb{P}_o(D \cap C^c) \leq c \log(m) m^{d(1-(\delta-1+\gamma)/(\gamma\delta))}$$

- *Lower bound* in Theorem II:
  - Strategy: Find vertex $|x| \leq m^{1/d}$, $R_x \approx m^{1/d}$ then $x \sim o$ and x has m neighbors
- *Upper bound* in Theorem II:
  - Proof for *scale-free percolation* $(R_x R_y)$: more edges than soft Boolean model
  - Coupling with *multitype branching process* only increases $\mathcal{N}_\beta$
  - $\mathbb{E}\mathcal{N}_\beta < \infty$ for $\gamma < 1/2$
  - Due to product structure of kernel: reduce to *single-type branching process* with mixed Poisson offspring distribution with parameter $\sim$ Pareto$(1/\gamma)$
  - $\mathcal{N}_\beta$ dominated by *total progeny* of branching process with offsprings $\sim$ Pareto$(1/\gamma)$
  - *Dwass' Theorem:* $\mathbb{P}_o(\mathcal{N}_\beta > m) \leq m^{1-1/\gamma}$

Thank you.