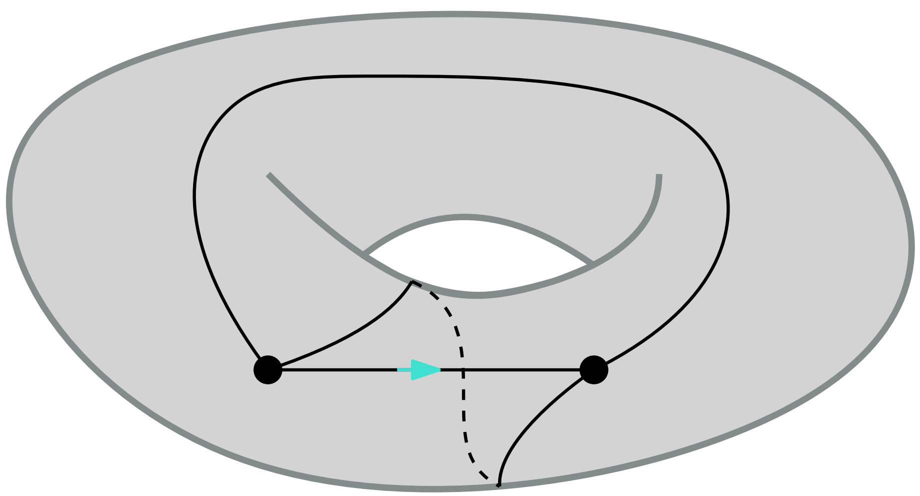
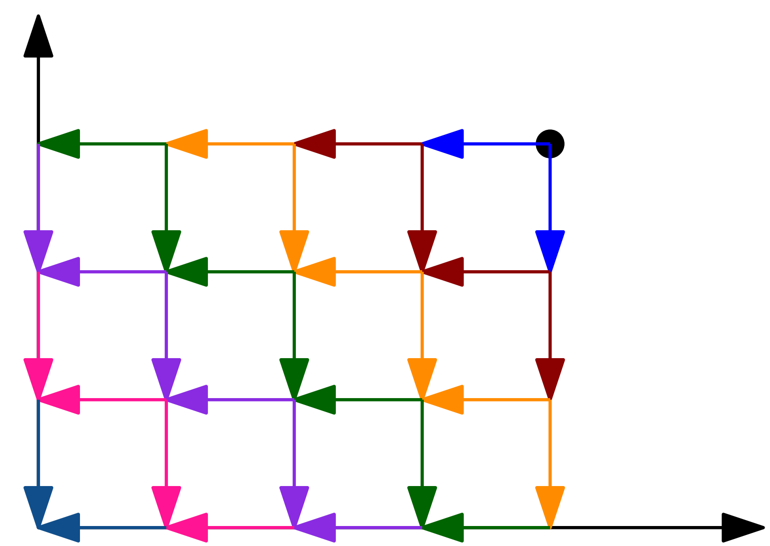
# Counting with random walks

Baptiste Louf
with Andrew Elvey–Price, Wenjie Fang and Michael Wallner





# 2-year postdoc position with me

Bordeaux, starting Fall 2025, deadline 31 December 2024

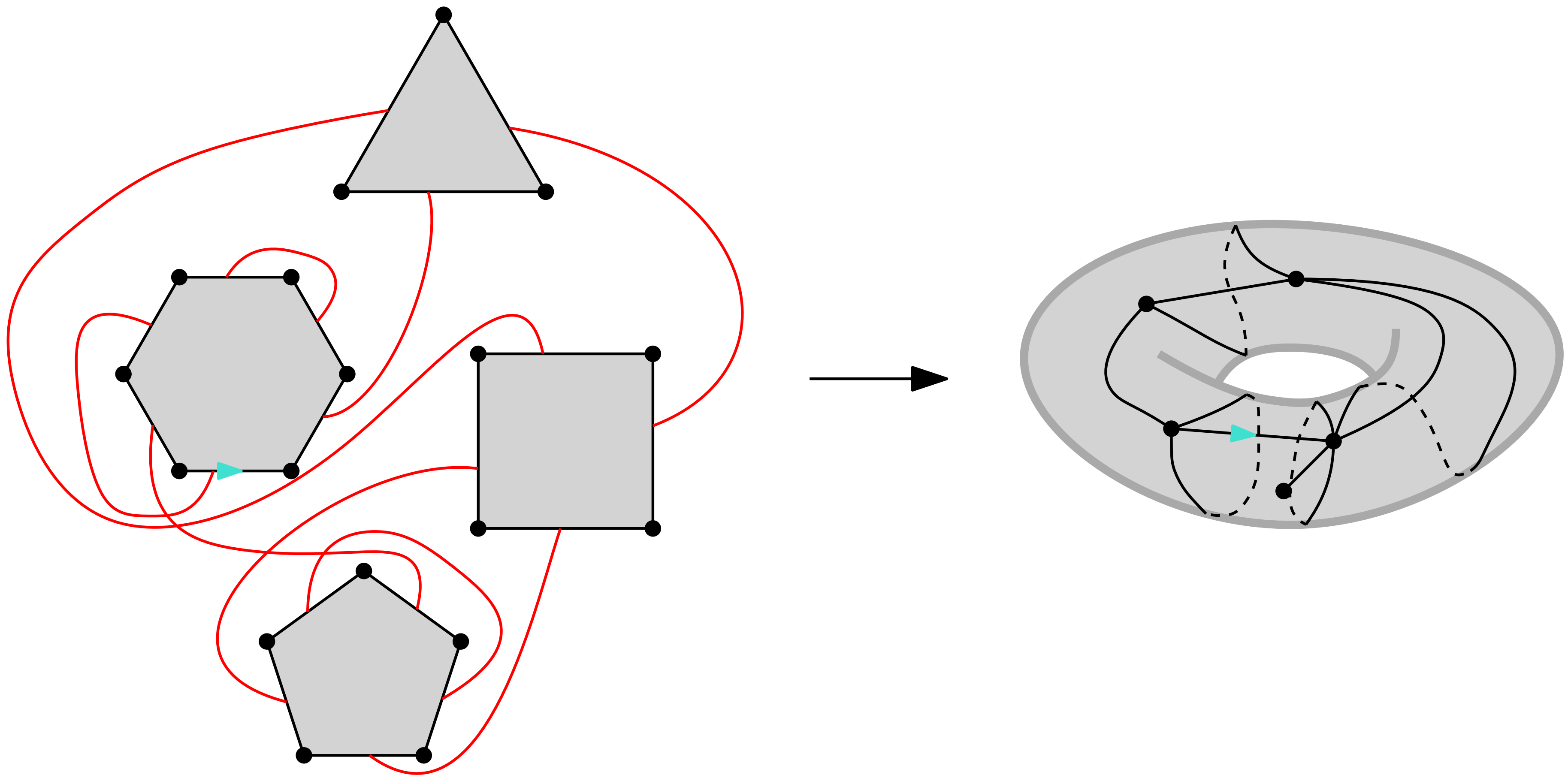
On **High genus geometry and/or asympotic enumeration**:
- hyperbolic geometry
- maps
- random graphs
- enumeration (cf this talk)
- and more !

For more info, check my website: **https://baptiste.louf.fr/**

# A bit of context:
# combinatorial maps and enumerative combinatorics

# Definition : maps

Map = **discrete surfaces**
i.e. gluing of polygons along their edges to create a (compact, connected, oriented) surface
Genus $g$ of the map = genus of the surface = # of handles



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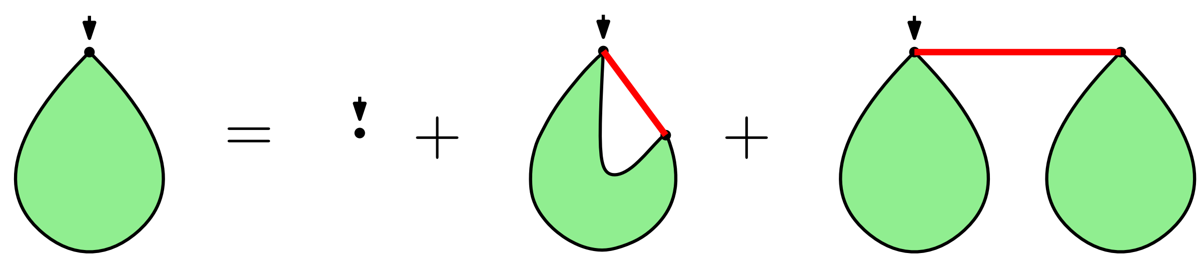
**Method :** Generating function and recursive decomposition

$$F(z) = \sum_{n \geq 0} a_n z^n$$



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**Theorem** [Lehman–Walsh '72+ Bender–Canfield '86] :
$a_n^{(g)} =$ nb of maps of genus g with $n$ edges

$$a_n^{(g)} \sim C_g 12^n n^{5/2(g-1)}$$

as $n \to \infty$ for $g$ fixed.

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**Large genus geometry:** negative average discrete curvature, **hyperbolic** behaviour.

# Bivariate asymptotics

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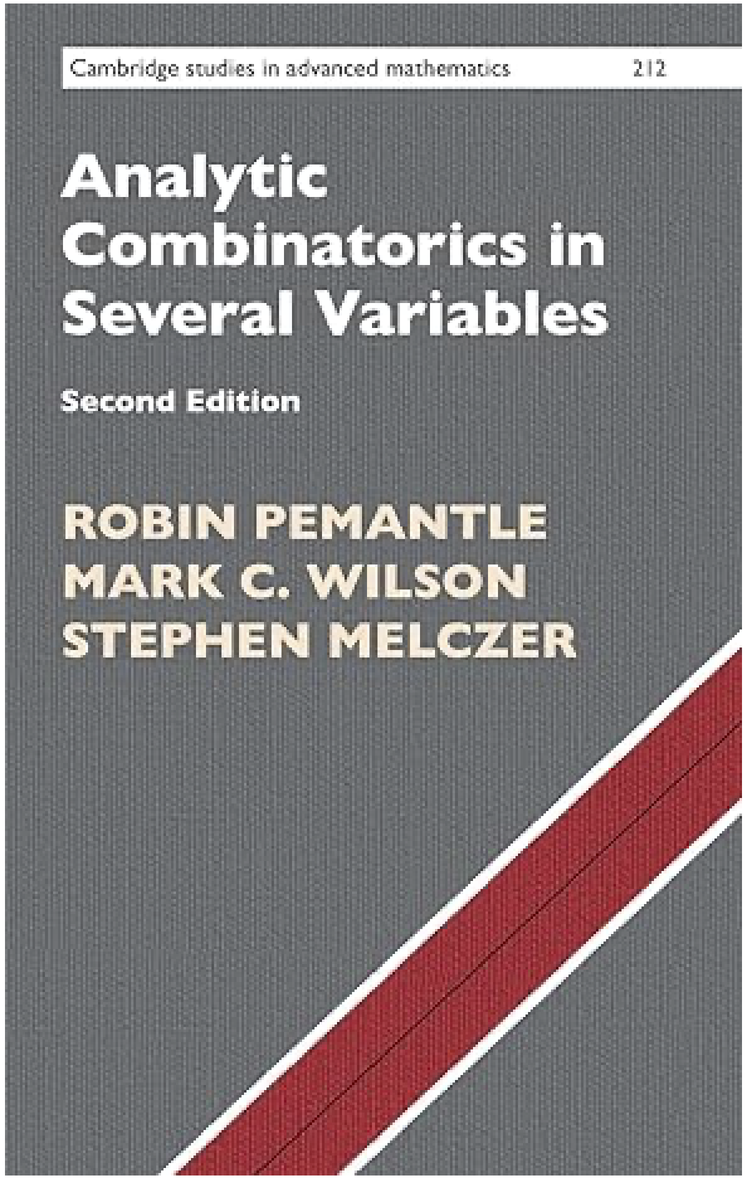


Recent progress to get asymptotics when the generating function is explicit and "simple"

(2nd edition, 2024, 550 pages !)

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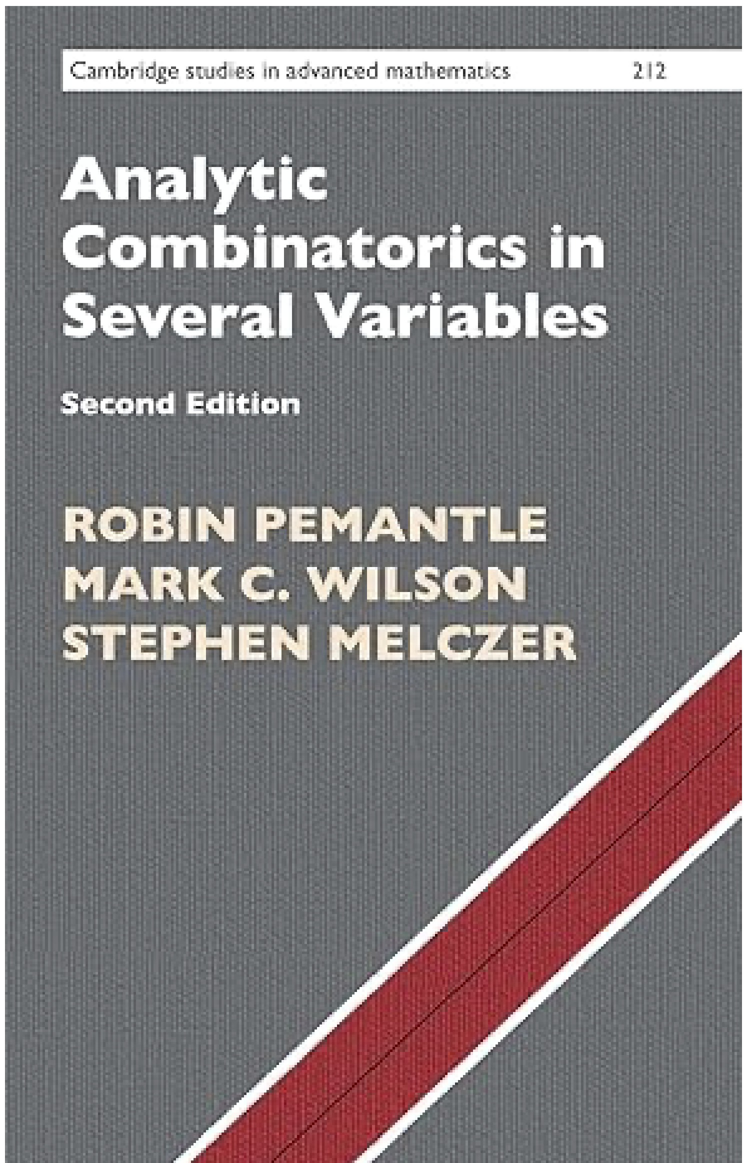


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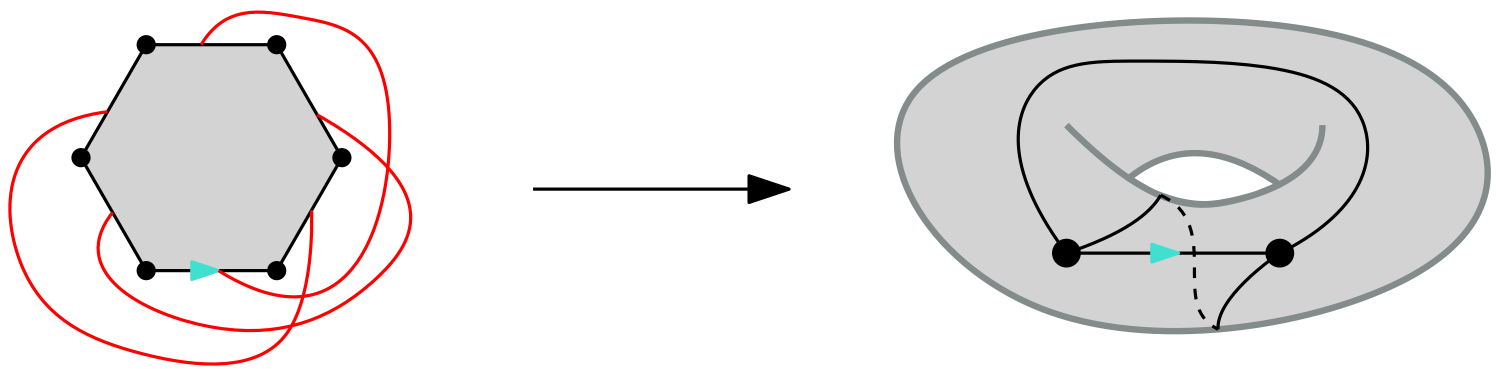
$\to$ maps do not fit in this case !

## The problem: enumerating unicellular maps, asymptotically, bivariately

# Unicellular maps

**Simplest model of maps:** maps with only one face/gluing of a single polygon



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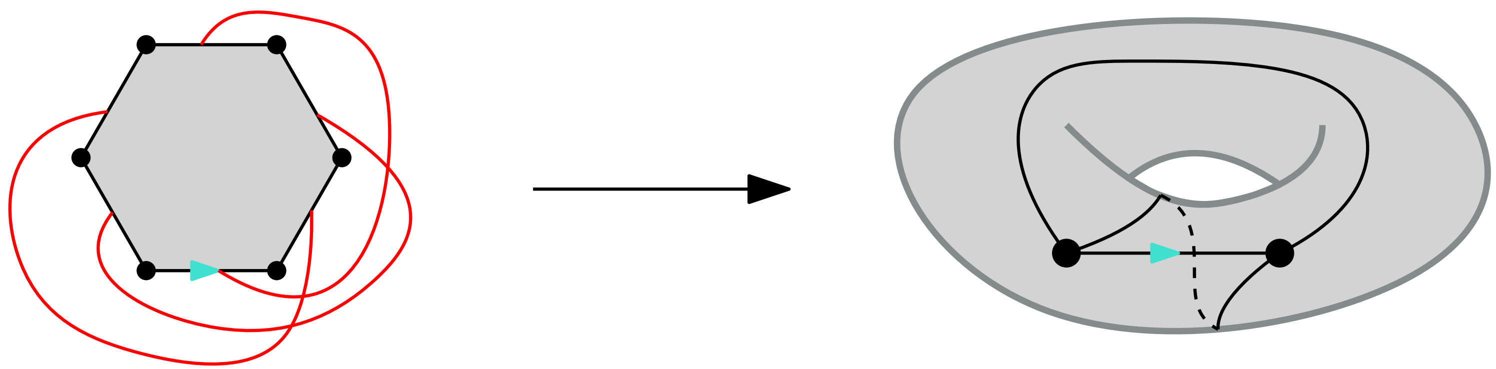

(unicellular map of genus $0$ = tree !)

Let $E(n, g)$ be the number of unicellular maps with $n$ edges and genus $g$
**Goal:** Study the asymptotics of $E(n, g)$ as $n, g \to \infty$ !

# Unicellular maps: what's known ?

**Theorem** [Harer–Zagier '86]: $E(0,0)=1$, for $n \geq 1, n \geq 2g$, we have

$$(n+1)E(n,g) = 2(2n-1)E(n-1,g) + (n-1)(2n-3)(2n-1)E(n-2,g-1)$$

$$\implies 1 + 2xy + 2 \sum_{g \geq 0, n > 0} \frac{E(n,g)}{(2n-1)!!} y^{n+1} x^{n+1-2g} = \left(\frac{1+y}{1-y}\right)^{x}$$

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**Asymptotic enumeration**:

- for $\frac{g}{n}\to\theta\in(0,1/2)$ [Angel–Chapuy–Curien–Ray '13]
- for $g=O(n^{1/3})$ [Curien–Kortchemski–Marzouk '23]

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**Method :** A bijection between unicellular maps and decorated trees [Chapuy–Féray–Fusy '12] (first case)
core/kernel decomposition (second case)

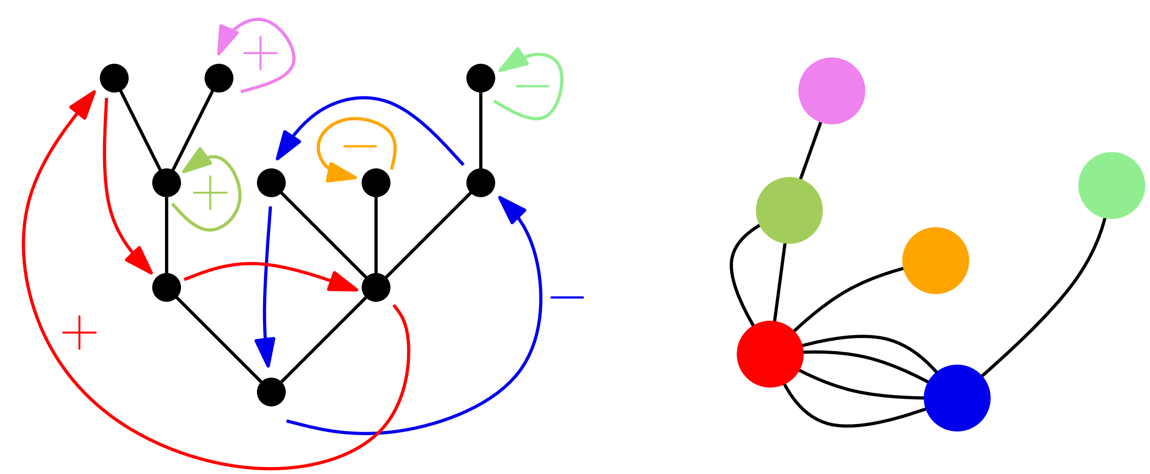


image : G. Chapuy

# Univellular maps: full asymptotics

**Our goal:** Obtain asymptotics for $E(n,g)$ for all regimes of $n, g$ using **only** the Harer-Zagier recurrence (we forget about the combinatorics !)

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**Theorem:** [Elvey-Price–Fang–L.–Wallner '2x]
As $n,g \to \infty$ with $n-2g >> \log(n)$

$$E(n,g) \sim \frac{1}{2\sqrt{2}\pi} n^{2g-2} e^{nf(\frac{g}{n})} J\left(\frac{g}{n}\right),$$

with

$$\theta(\lambda) = \frac{1}{2} - \frac{\lambda \log\left(\frac{1+\sqrt{1-4\lambda}}{1-\sqrt{1-4\lambda}}\right)}{\sqrt{1-4\lambda}},$$

$$f(\theta) = -\theta \log\left(\frac{1-4\lambda}{4\lambda^2}\right) - 2\theta - \log(\lambda),$$

$$J(\theta) = \sqrt{\frac{2}{\lambda(\theta)(1-4\lambda(\theta)-2\theta+4\theta\lambda(\theta))}}.$$

## Idea of proof 1: guess and check

## Guess and check

$$(\mathbf{HZ}) \quad (n+1)E(n,g) = 2(2n-1)E(n-1,g) + (n-1)(2n-3)(2n-1)E(n-2,g-1)$$

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$$\Omega(n,0) \sim E(n,0) \text{ as } n \to \infty \qquad \text{“asymptotic initial condition”}$$

$$(n+1)\Omega(n,g) \approx 2(2n-1)\Omega(n-1,g) + (n-1)(2n-3)(2n-1)\Omega(n-2,g-1) \qquad \text{“asymptotic recurrence”}$$

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"asymptotic recurrence"

Then hopefully

$$\Omega(n,g) \sim E(n,g)$$

# Idea of proof 2: random walks

## Rewriting the recurrence in terms of walks

Set $A(v,g) := E(v+2g,g)$ Harer–Zagier rewrites

$$A(\mathbf{v},\mathbf{g}) = \frac{2(2n-1)}{n+1}A(\mathbf{v}-\mathbf{1},\mathbf{g}) + \frac{(n-1)(2n-3)(2n-1)}{n+1}A(\mathbf{v},\mathbf{g}-\mathbf{1})$$

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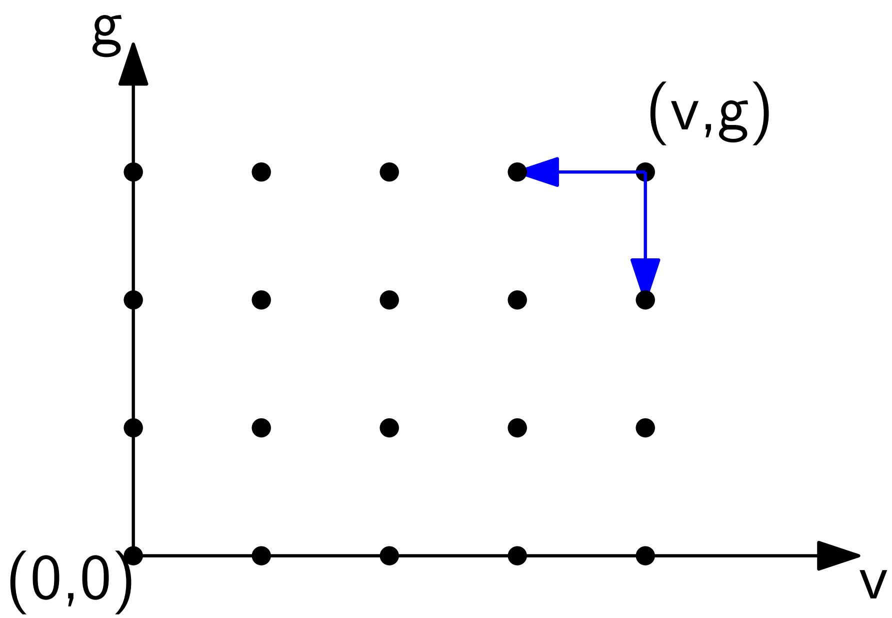

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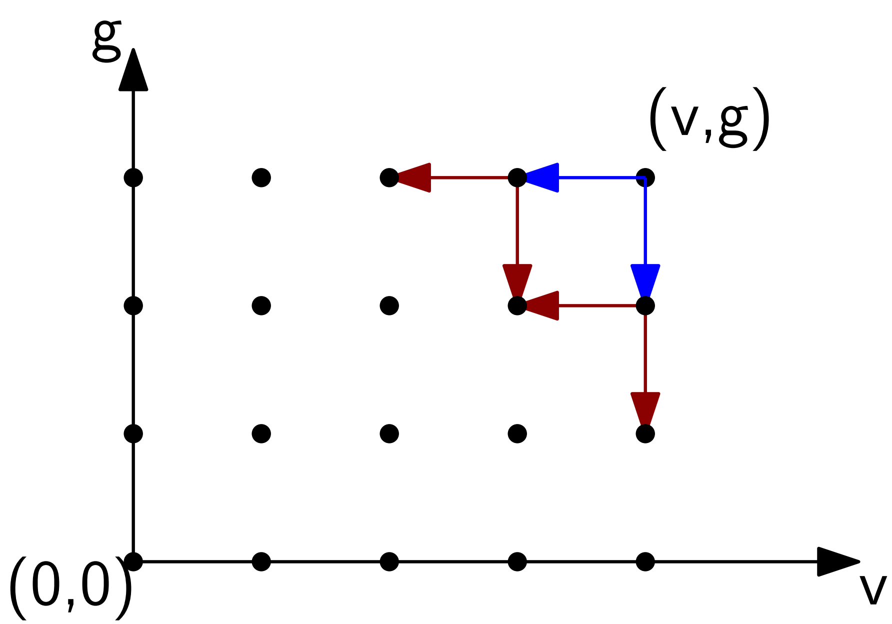


$$\begin{aligned}
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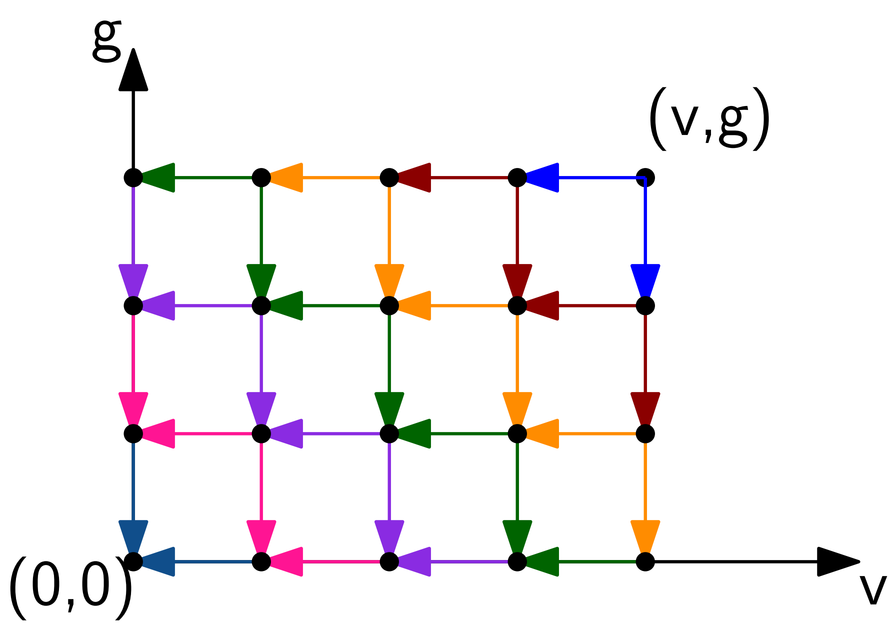


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\end{aligned}$$

$$= \sum_{\text{paths from } (v,g) \text{ to } (0,0)} \prod_{\text{steps of the paths}} weight(step)$$

(because $A(0,0) = 1$ !)

# Modelling by random walks: first ideas

**Question:** What are the paths that contribute to the counting ?
Behaviour of RW started from $N_0, G_0$, with weight steps:

$$\frac{2(2n-1)}{n+1}\frac{E(n-1,g)}{E(n,g)} \quad \text{and} \quad \frac{(n-1)(2n-3)(2n-1)}{n+1}\frac{E(n-2,g-1)}{E(n,g)}$$

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**Approximation goal:** Find $\Omega(n,g)$ such that

$$\frac{2(2n-1)}{n+1}\frac{\Omega(n-1,g)}{\Omega(n,g)} + \frac{(n-1)(2n-3)(2n-1)}{n+1}\frac{\Omega(n-2,g-1)}{\Omega(n,g)} \approx 1$$

# Proof: more details

# Defining $\Omega(n,g)$

**Setup:**

$$\Omega(n,g) := \frac{1}{2\sqrt{\pi}} \frac{\sqrt{g}(g/e)^g}{g!} n^{2g-2} e^{nf\left(\frac{g}{n}\right)} J\left(\frac{g}{n}\right) \frac{\sqrt{2\pi}(n-2g)^{n-2g+1}}{e^{(n-2g)}\Gamma(n-2g+3/2)},$$

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← **summable !**

This means that our approximation by a random walk will be valid !

# Defining the random walk

**Setup:** Start from $N_0, G_0 = n, g$, stop when $G_k = 0$ or $N_k = 2G_k$.
Stopping time $\tau = \tau(n, g)$

$$(N_{k+1}, G_{k+1}) = (N_k - 1, G_k) \quad \text{with proba } \frac{\alpha(N_k, G_k)}{\alpha(N_k, G_k) + \beta(N_k, G_k)}$$

$$(N_{k+1}, G_{k+1}) = (N_k - 2, G_k - 1) \quad \text{with proba } \frac{\beta(N_k, G_k)}{\alpha(N_k, G_k) + \beta(N_k, G_k)}$$

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**Conserved quantity:**

$$Q(n, g) := \frac{E(n, g)}{\Omega(n, g)}$$

**(HZ)** rewrites

$$Q(n, g) = \alpha(n, g)Q(n - 1, g) + \beta(n, g)Q(n - 2, g - 1)$$

Hence

$$\mathbb{E}(Q(N_{k+1}, G_{k+1})) \approx \mathbb{E}(Q(N_k, G_k))$$

# Typical behaviour and asymptotic result

**Typical behaviour:**

**Propostion:** As $n, g \to \infty$ with $n - 2g >> \log n$, with "very high probability":

$$G_\tau = 0 \quad \text{and} \quad N_\tau \to \infty$$

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## Asymptotics as a corollary:

Since $Q(n, 0) \to 1$ as $n \to \infty$,

$$\mathbb{E}(Q(N_\tau, G_\tau)) \sim 1$$

, but

$$\mathbb{E}(Q(N_\tau, G_\tau)) \sim Q(N_0, G_0) = Q(n, g)$$

hence

$$E(n, g) \sim \Omega(n, g)$$

## How to guess ?

$$(\mathbf{HZ}) \quad (n+1)E(n,g) = 2(2n-1)E(n-1,g) + (n-1)(2n-3)(2n-1)E(n-2,g-1)$$

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**Plotting:**

Fix $\theta \in (0, 1/2)$ and plot

$$\frac{E(\lfloor \theta^{-1} g \rfloor, g-1)}{E(\lfloor \theta^{-1} g \rfloor, g)}$$ it grows like $g^2$

$$\frac{E(\lfloor \theta^{-1} g \rfloor - 1, g)}{E(\lfloor \theta^{-1} g \rfloor, g)}$$ it converges but depends on $\theta$

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**"Guess and check":**

$$E(n,g) \approx n^{2g} exp\left(n f(g/n)\right)$$

First order of **(HZ)** gives a differential equation for $f$.

# Conclusion

**Recap:** Approximating a linear bivariate recurrence by a random walk
*Guess and check* type of method
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- sticky walls $\rightarrow$ bouncy walls
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**Other works on recurrences and random walks:**
[Aggarwal '18,'20, Elvey-Price–Fang–Wallner '19,'20, Chassaing–Flin '22,...]

Thank you !