Random (Laguerre) tessellations

Anna Gusakova - Münster University
(joint work with Zakhar Kabluchko, Christoph Thäle, Mathias in Wolde-Lübke)

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Basic definitions: Tessellations

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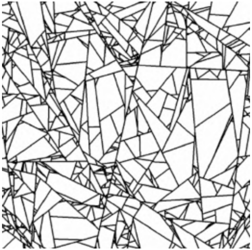

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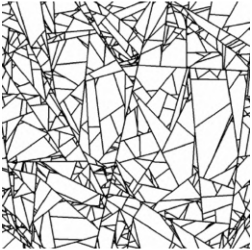

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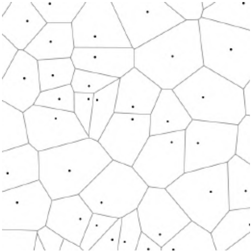

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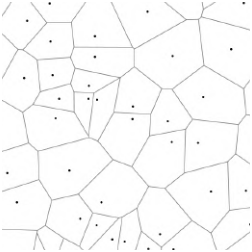

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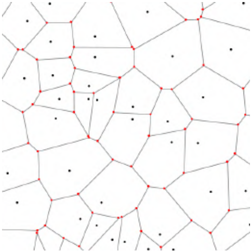

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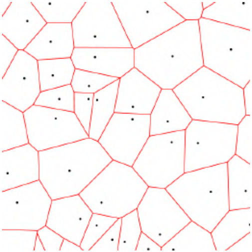

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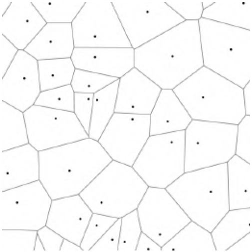

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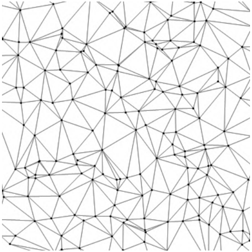

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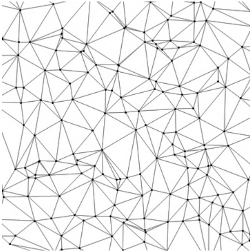

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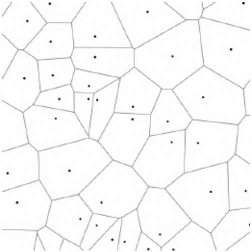

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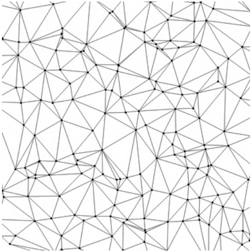

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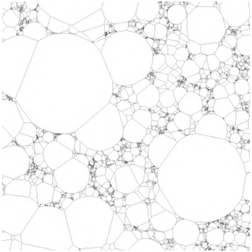

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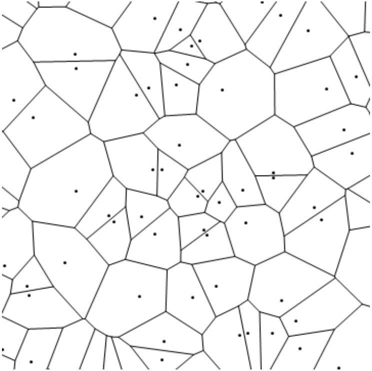

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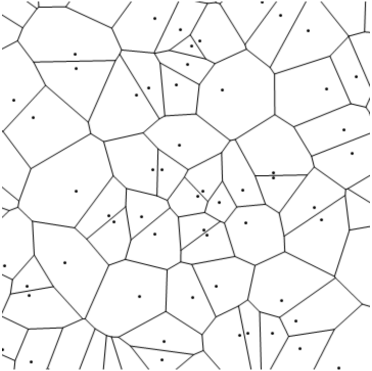

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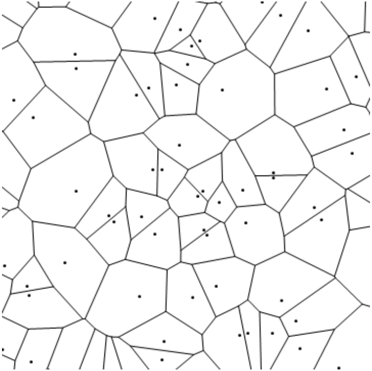

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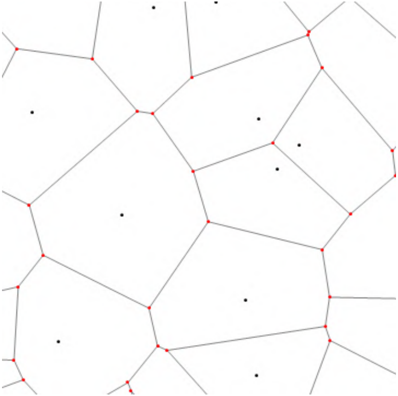
1950's - current: Baumstark, Blaszczyszyn, Calka, Hug, Kendall, Last, Møller, Mecke, Miles, Muche, Reitzner, Schneider, Stoyan, Zhang and many others.

Poisson-Delaunay tessellation (definition via duality)



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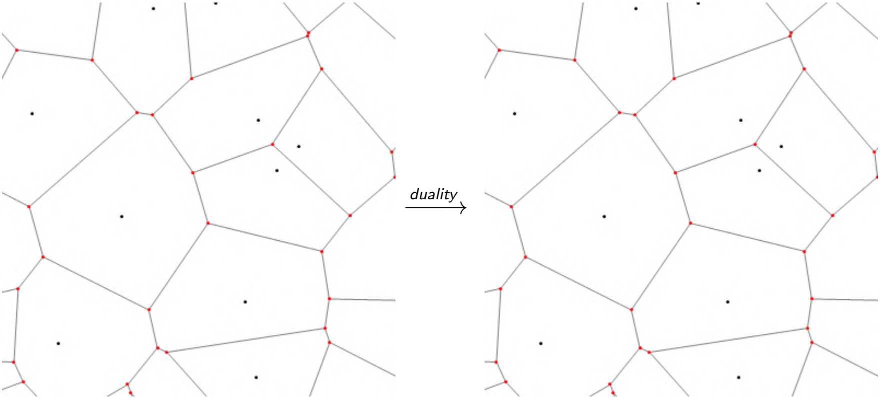

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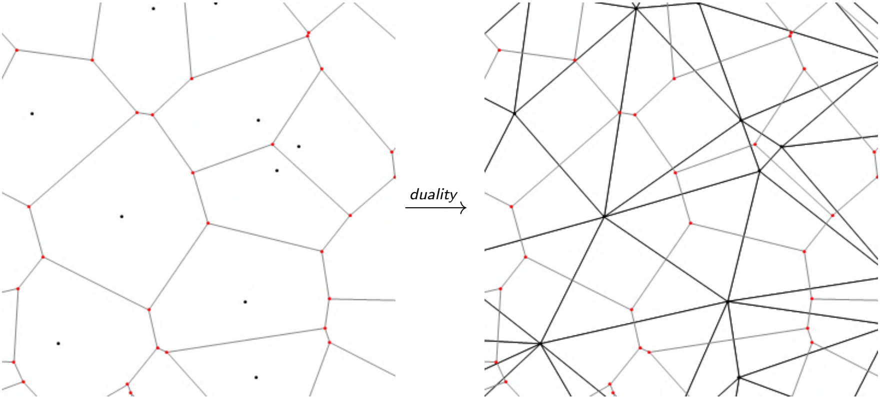

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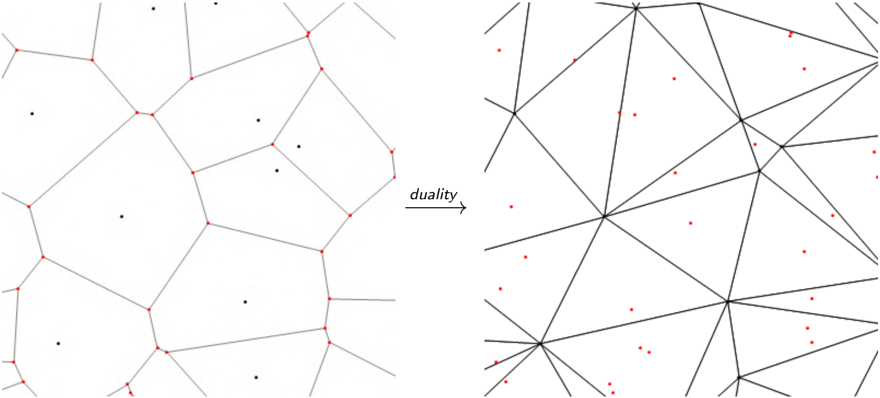

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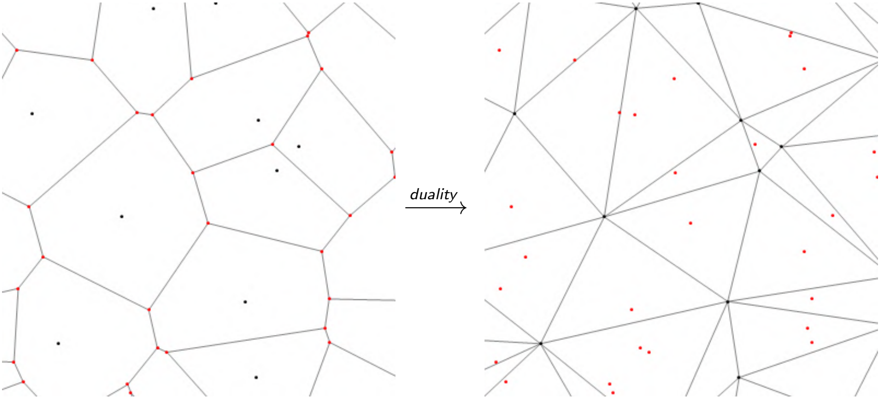

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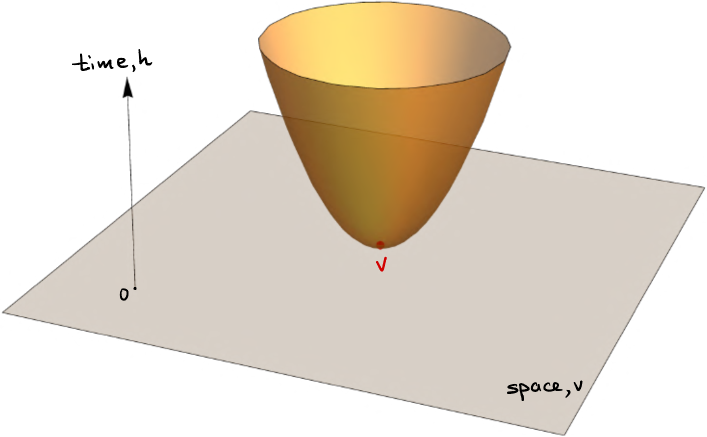

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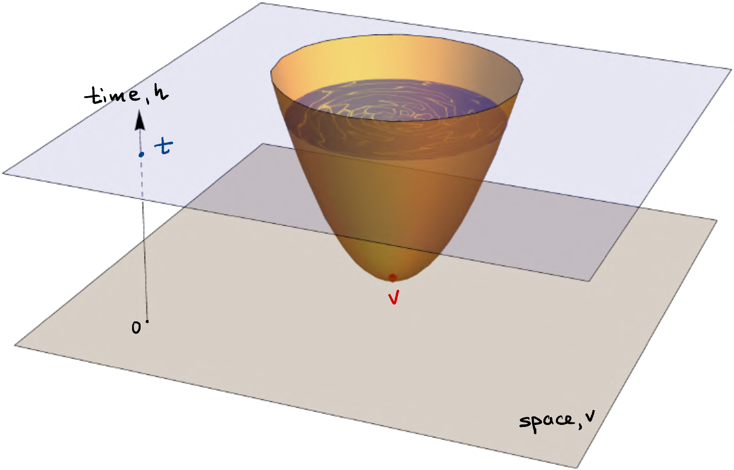

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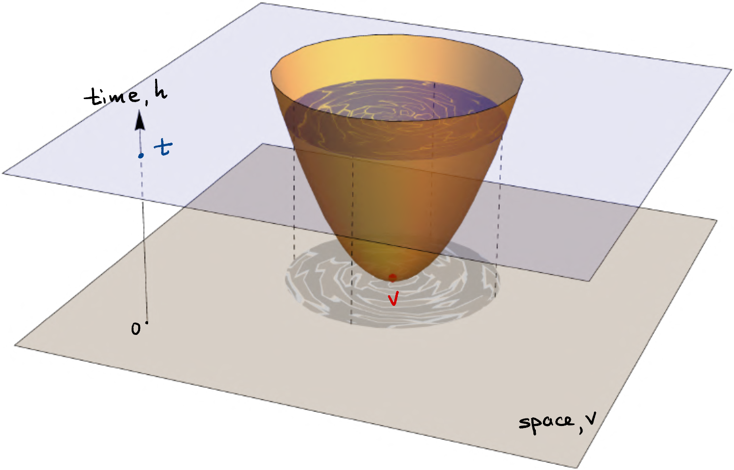

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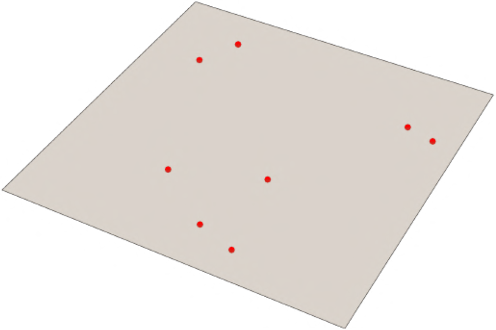

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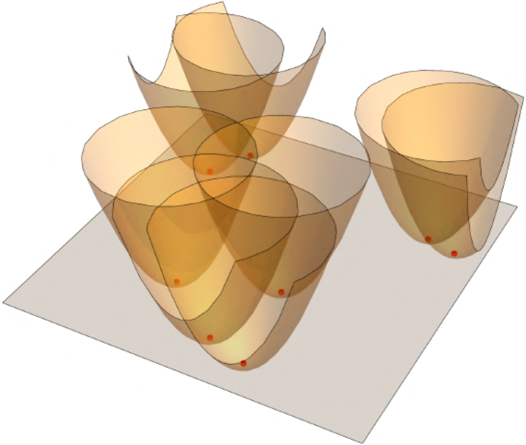

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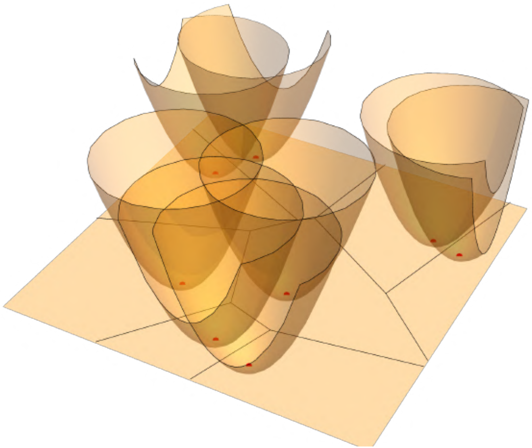

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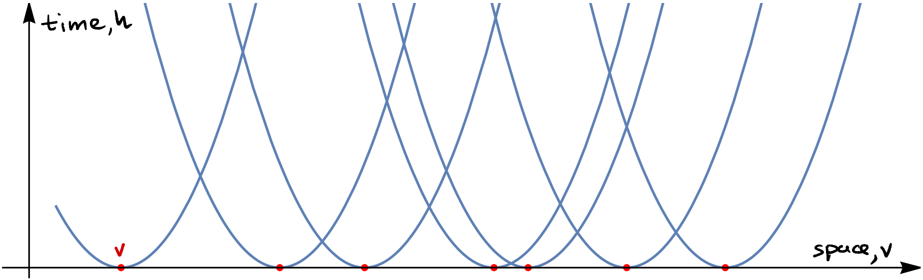

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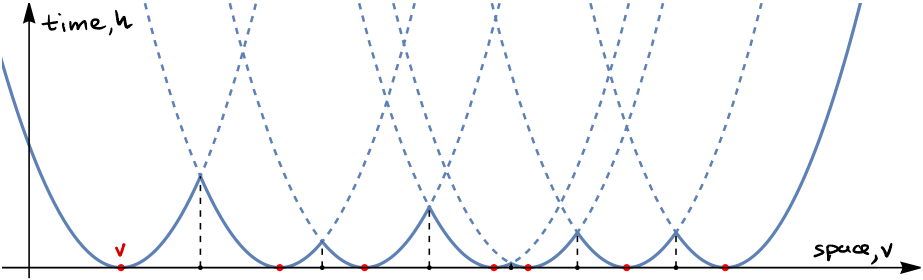

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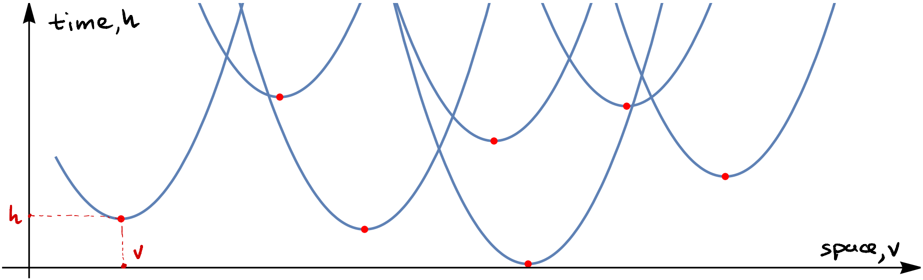

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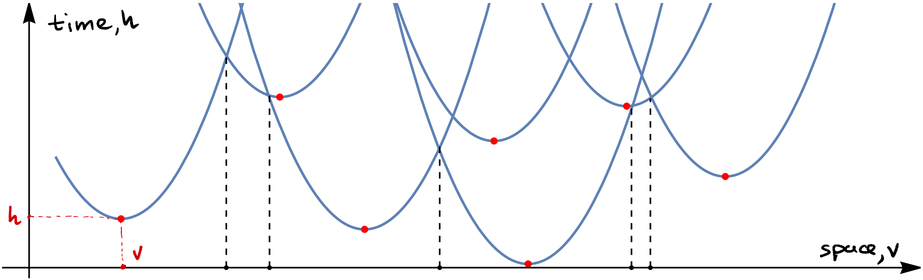

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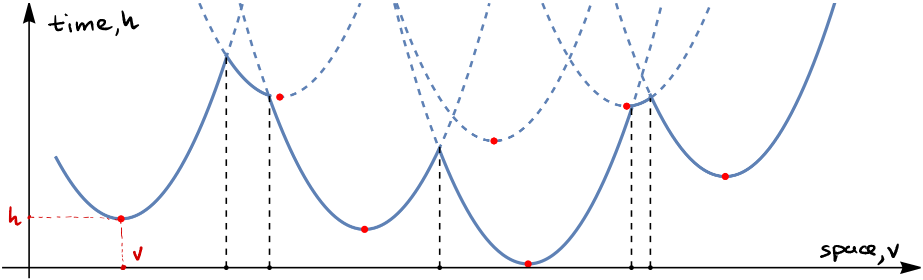
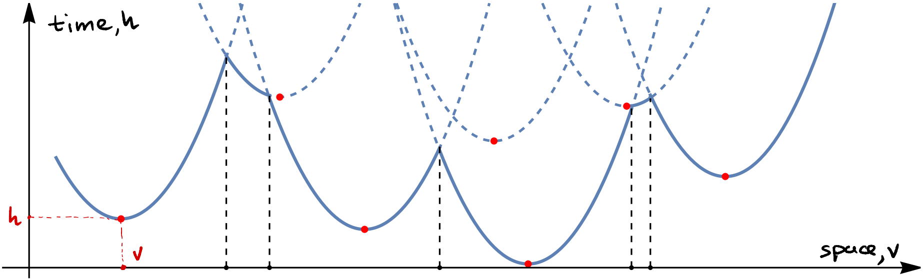


Poisson-Voronoi tessellation: graphical interpretation

Generalized Voronoi (Laguerre) cell of $(v, h) \in \xi$:

$$V((v,h),\xi) := \{z \in \mathbb{R}^d \colon \|z - v\|^2 + h \leq \|z - v'\|^2 + h' \text{ for all } (v', h') \in \xi\}.$$



Motivation: sectional properties of Poisson-Voronoi tessellation

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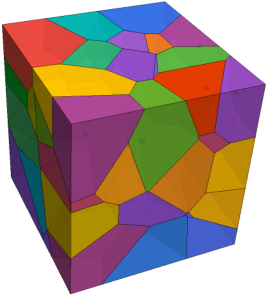
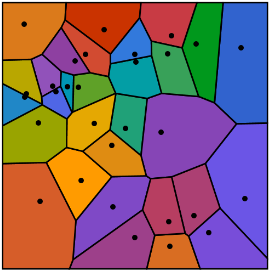
Question: Is sectional Poisson-Voronoi tessellation $\mathcal{V}_d \cap L_\ell$ a Voronoi tessellation?



3D Poisson-Voronoi tessellation



2D Poisson-Voronoi tessellation

Motivation: sectional properties of Poisson-Voronoi tessellation

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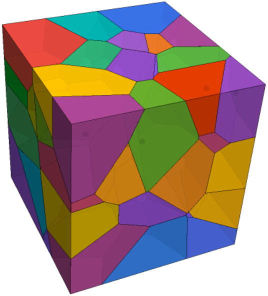

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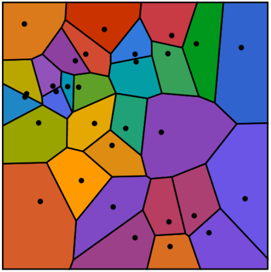

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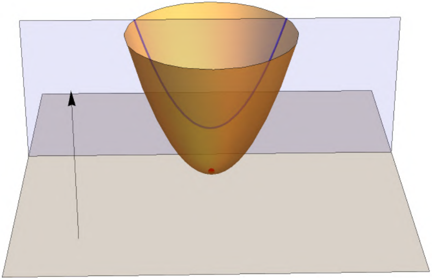

Answer: "The sectional Poisson-Voronoi tessellation is not a Voronoi tessellation" (Chiu, Van De Weygaert, Stoyan'96). It is a Laguerre tessellation!

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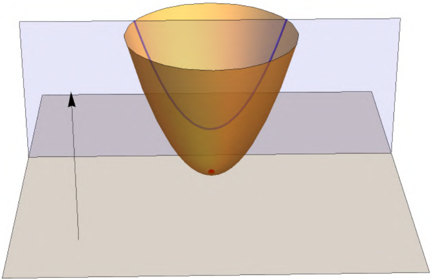
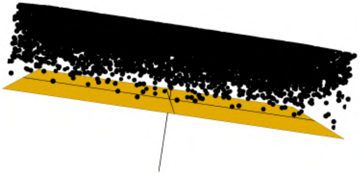


Answer: "The sectional Poisson-Voronoi tessellation is not a Voronoi tessellation" (Chiu, Van De Weygaert, Stoyan'96). It is a Laguerre tessellation! But which one?

Generalized Voronoi tessellation (Laguerre tessellation)

Let ξ_f be a PPP in $\mathbb{R}^d \times E$ where E is $\mathbb{R}$, $\mathbb{R}_{\geq 0}$ or $\mathbb{R}_{<0}$ with intensity measure

$$\mu(B \times A) = \int_{\mathbb{R}^d} \int_E \mathbf{1}\{v \in B\}\mathbf{1}\{h \in A\}f(h)\mathrm{d}v\mathrm{d}h.$$



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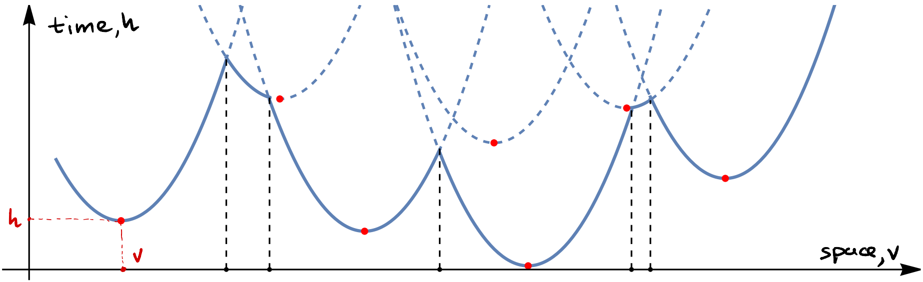

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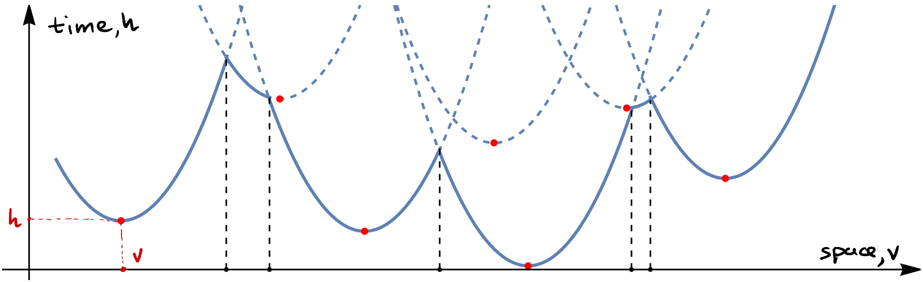

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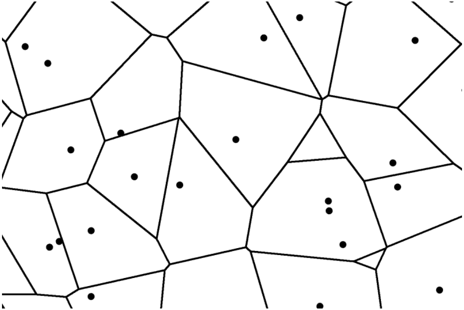

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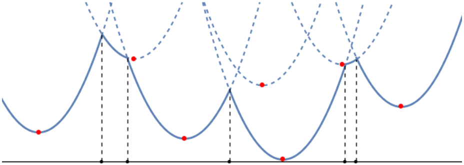

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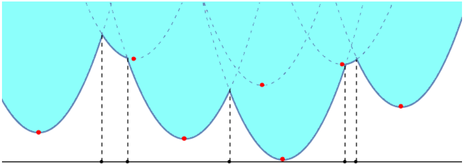


$$\mathbb{P}((v,h) \notin \Psi(f)) = \exp(-(I^{\frac{d}{2}+1} f)(h))$$

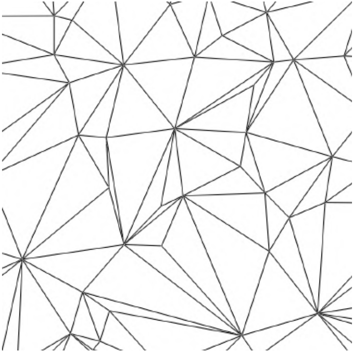
Examples

- If $f \in L^1(\mathbb{R})$, then ξ_f is an independent marking of a homogeneous PPP on $\mathbb{R}^d$ (Lautensack, Zuyev'08).

Examples

- Gaussian model: take $f(h) := e^h$

  $\widetilde{\mathcal{V}}_d := \mathcal{L}_d(f)$ is called Gaussian-Voronoi tessellation;
  $\widetilde{\mathcal{D}}_d := (\mathcal{L}_d(f))^*$ is called Gaussian-Delaunay tessellation.



Examples

- β-model: $f(h) := c_{d,\beta}\, h^{\beta} \mathbf{1}_{h \geq 0}$, $\beta > -1$

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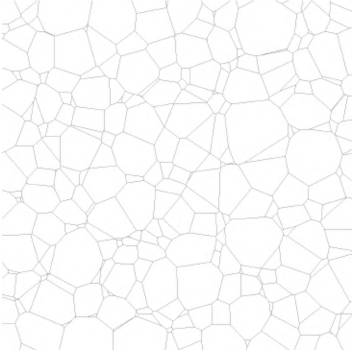

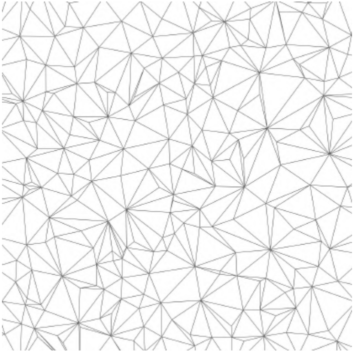

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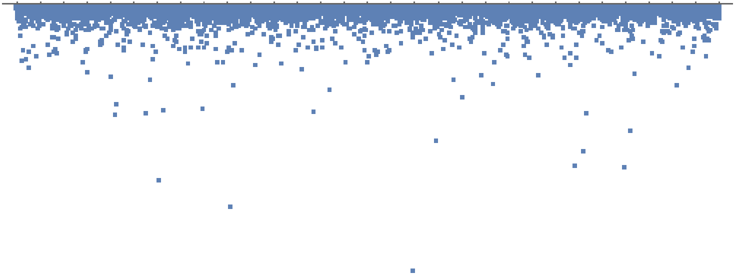

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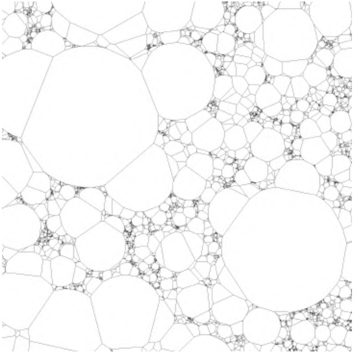

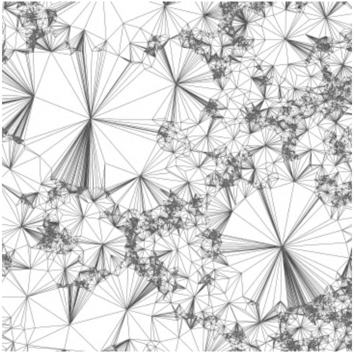

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Question: Is sectional Poisson-Voronoi tessellation $\mathcal{V}_d \cap L_\ell$ a Voronoi tessellation?

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For any ℓ-dimensional affine subspace L_ℓ we have

$$\mathcal{V}_d \cap L_\ell \overset{d}{=} \mathcal{V}_{\ell,-1+(d-\ell)/2} \qquad \textit{(up to isometry)}$$

(here intensity of $\mathcal{V}_d$ is chosen to be $\gamma = \pi^{\frac{d+1}{2}}/\Gamma(\frac{d+1}{2})$)

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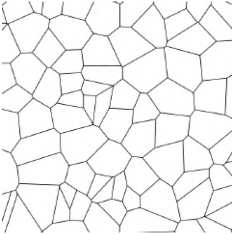
As a corollary:

$$\mathcal{V}_{d,\beta} \cap L_\ell \stackrel{d}{=} \mathcal{V}_{\ell,\beta+(d-\ell)/2} \qquad \mathcal{V}'_{d,\beta} \cap L_\ell \stackrel{d}{=} \mathcal{V}'_{\ell,\beta-(d-\ell)/2} \qquad \widetilde{\mathcal{V}}_d \cap L_\ell \stackrel{d}{=} \widetilde{\mathcal{V}}_\ell$$

Characteristics of stationary random tessellation

- $\mathcal{T}$ is a stationary random tessellation.



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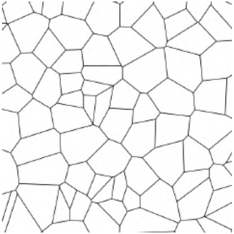

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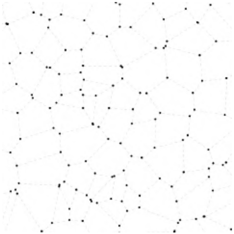

$k = 0$

Characteristics of stationary random tessellation

- $\mathcal{T}$ is a stationary random tessellation.
- k-**cell intensity of** $\mathcal{T}$:

$$\gamma_k(\mathcal{T}) = \mathbb{E} \sum_{F \in \mathcal{F}_k(\mathcal{T})} \mathbf{1}\{z(F, \mathcal{T}) \in [0,1]^d\},$$

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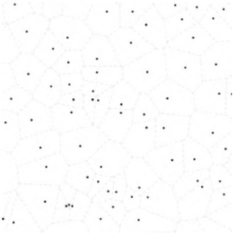

$k = 2$

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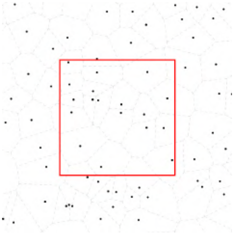

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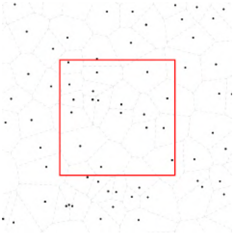

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- **Typical cell of** $\mathcal{T}$: $Z(\mathcal{T})$ is a random polytope with distribution

$$\mathbb{P}_{\mathcal{T}}^{\text{typ}}(\cdot) = \frac{1}{\gamma_d(\mathcal{T})} \mathbb{E} \sum_{t \in \mathcal{T}:\ z(t,\mathcal{T}) \in [0,1]^d} \mathbf{1}\{t - z(t, \mathcal{T}) \in \cdot\}.$$

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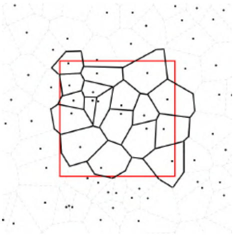

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Stochastic representation of the typical cell

Let $\mathcal{T}$ be one of the following tessellations $\mathcal{D}_d$, $\mathcal{D}_{d,\beta}$, $\mathcal{D}'_{d,\beta}$, $\widetilde{\mathcal{D}}_d$.

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Theorem (A.G., Z. Kabluchko and C.Thäle, 2022)

We have $Z(\mathcal{T}) \overset{d}{=} R \cdot \mathrm{conv}(Y_1, \dots, Y_{d+1}) := \mathrm{conv}(RY_1, \dots, RY_{d+1})$;

(a) $(Y_1, \dots, Y_{d+1})$ *are random points, whose joint distribution is*

$$const \cdot \mathrm{Vol}(\mathrm{conv}(y_1, \dots, y_{d+1})) \prod_{i=1}^{d+1} F(\mathrm{d}y_i);$$

(b) R *is a random variable on* $(0, \infty)$ *with distribution* G;

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$$F = \operatorname{Unif}(\mathbb{S}^{d-1}), \qquad G(\mathrm{d}r) = r^{d^2-1} e^{-c \cdot r^d} \mathrm{d}r, \qquad \mathcal{T} = \mathcal{D}_d;$$

$$F(\mathrm{d}x) = c_{d,\beta}(1 - \|x\|^2)^\beta \mathbf{1}_{\mathbb{B}^d}(\|x\|)\mathrm{d}x, \qquad G(\mathrm{d}r) = r^{(d+1)^2+2(d+1)\beta} e^{-c \cdot r^{d+2+2\beta}} \mathrm{d}r, \qquad \mathcal{T} = \mathcal{D}_{d,\beta};$$

$$F(\mathrm{d}x) = c'_{d,\beta}(1 + \|x\|^2)^{-\beta} \mathrm{d}x, \qquad G(\mathrm{d}r) = r^{(d+1)^2-2(d+1)\beta} e^{-c \cdot r^{d+2-2\beta}} \mathrm{d}r \qquad \mathcal{T} = \mathcal{D}'_{d,\beta};$$

$$F(\mathrm{d}x) = (2\pi)^{-d/2} e^{-\|x\|^2/2} \mathrm{d}x, \qquad G = \delta_1, \qquad \mathcal{T} = \widetilde{\mathcal{D}}_d.$$

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Remark

Case of Poisson-Delaunay tessellation $\mathcal{T} = \mathcal{D}_d$ *is well-know (Miles'74, Møller'94).*

New formulas for the classical Poisson-Voronoi tessellation

Combining combinatorial relations between different characteristics of stationary random tessellation and properties of β-random simplices we also get:

Theorem (A.G., Z. Kabluchko and C. Thäle, 2024)

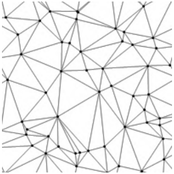
For any $0 \leq j \leq d$ *we have*

$$\mathbb{E} V_j(Z(\mathcal{V}_d)) = \left(\frac{\gamma}{\Gamma(\frac{d}{2}+1)}\right)^{-\frac{j}{d}} \frac{(d-j+1)(d-1)+1}{\sqrt{\pi}\, d(d-j)!} \frac{\Gamma(d-j+\frac{j}{d})}{\Gamma(\frac{j+1}{2})} \times \int_{-\infty}^{\infty} (\cosh u)^{-(d-j+1)(d-1)-2} \left[\frac{\sqrt{\pi}\,\Gamma(\frac{d}{2}+1)}{\Gamma(\frac{d+1}{2})} + d\,\mathrm{i} \int_0^u (\cosh v)^{d-1} \mathrm{d}v\right]^{d-j} \mathrm{d}u.$$

V_j is j-th intrinsic volume, in particular V_d is the volume and $\frac{1}{2} V_{d-1}$ is the surface area.

Connection between the models

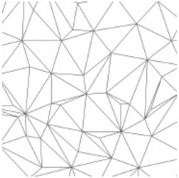
Classical models: $\mathcal{V}_d$, $\mathcal{D}_d$



Homogeneous PPP in $\mathbb{R}^d$, e.g.

$$f = \delta_0.$$

β-models: $\mathcal{V}_{d,\beta}$, $\mathcal{D}_{d,\beta}$



$$f(h) = c_{d,\beta} h^{\beta},$$
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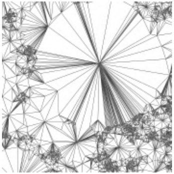

$$f(h) = c'_{d,\beta} h^{-\beta},$$
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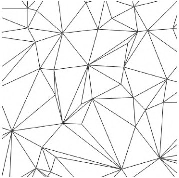
Gaussian-models: $\widetilde{\mathcal{V}}_d$, $\widetilde{\mathcal{D}}_d$



$$f(h) = e^h.$$

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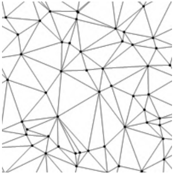

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$\xleftarrow{\beta \to -1}$

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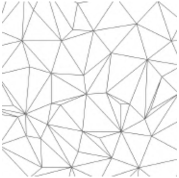

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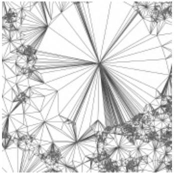

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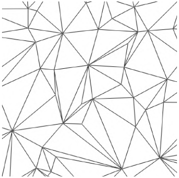
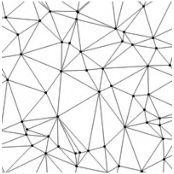


$$f(h) = e^{h}.$$

Connection between the models (convergence of typical cells)

Classical models: $\mathcal{V}_d$, $\mathcal{D}_d$



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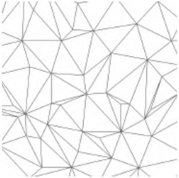


$f(h) = c_{d,\beta} h^{\beta}$,

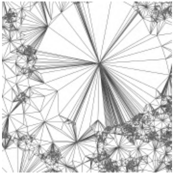
$h \geq 0$,

$\beta \in (-1, \infty)$.

$\xleftarrow{\beta \to -1}$ (from β-models to classical models)

$\downarrow$ $\beta \to \infty$ (when scaled by $\sqrt{2\beta}$) (from β-models to Gaussian-models)

β'-models: $\mathcal{V}'_{d,\beta}$, $\mathcal{D}'_{d,\beta}$



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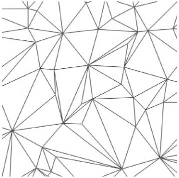
Gaussian-models: $\widetilde{\mathcal{V}}_d$, $\widetilde{\mathcal{D}}_d$



$f(h) = e^h$.

Connection between the models (convergence in a stronger sense)

Consider the skeleton of a stationary random tessellation $\mathcal{T}$:

$$\mathrm{skel}(\mathcal{T}) := \bigcup_{t \in \mathcal{T}} \mathrm{bd}\, t,$$

which is a stationary random closed set.

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$\mathscr{D}_{d,\beta}$ and $\mathscr{D}'_{d,\beta}$ converge weakly to $\widetilde{\mathscr{D}}_d$ as $\beta \to \infty$.

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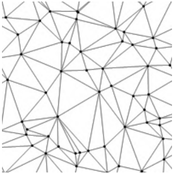
Remark: If it holds we could formally write $\mathcal{D}_{d,-1} := \mathcal{D}_d$ and allow $\beta \geq -1$ in β-model.

Connection between the models (convergence of skeletons)

Classical models: $\mathcal{V}_d$, $\mathcal{D}_d$



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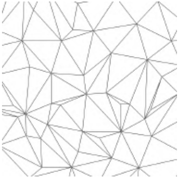

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β-models $\xrightarrow{\beta \to -1}$ Classical models

β-models $\xrightarrow{\beta \to \infty}$ Gaussian-models (when scaled by $\sqrt{2\beta}$)

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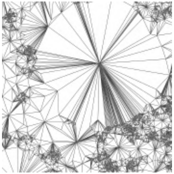

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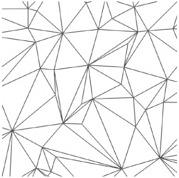

$$f(h) = e^h.$$

β'-models $\downarrow$ $\beta \to d/2+1$

Question: What makes then β-, β'- and Gaussian models so special?

Extreme value theory

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Fisher-Tippett-Gnedenko theorem: Let $X_1, X_2, \ldots$ be the sequence of i.i.d. random variables and let $M_n = \max_{1 \le i \le n} X_i$. Suppose there are $a_n > 0$, $b_n \in \mathbb{R}$, s.t.

$$\lim_{n \to \infty} \mathbb{P}((M_n - b_n) \le a_n x) = G(x),$$

for some non-degenerate distribution function G. Then G (after proper renormalization) is one of the following extreme value distributions

- **Weibull distribution:** $\Psi_\beta(t) = \exp(-(-t)^\beta)$, $t < 0$, $\beta > 0$;
- **Fréchet distribution:** $\Phi_\beta(t) = \exp(-t^{-\beta})$, $t > 0$, $\beta > 0$;
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Question: What are the high-dimensional settings?

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Let $X_1, X_2, \dots$ be i.i.d. random points in $\mathbb{R}^d$, whose distribution is rotationally invariant.

Extreme value theory

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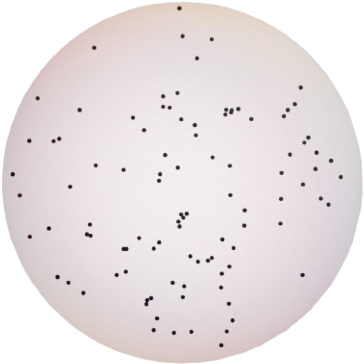
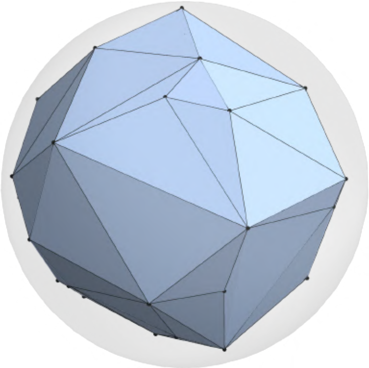
Question: What are the high-dimensional settings?

Let $X_1, X_2, \dots$ be i.i.d. random points in $\mathbb{R}^d$, whose distribution is rotationally invariant. Natural high-dimensional analogue of maximum is the convex hull $\mathrm{conv}(X_1, \dots, X_n)$.





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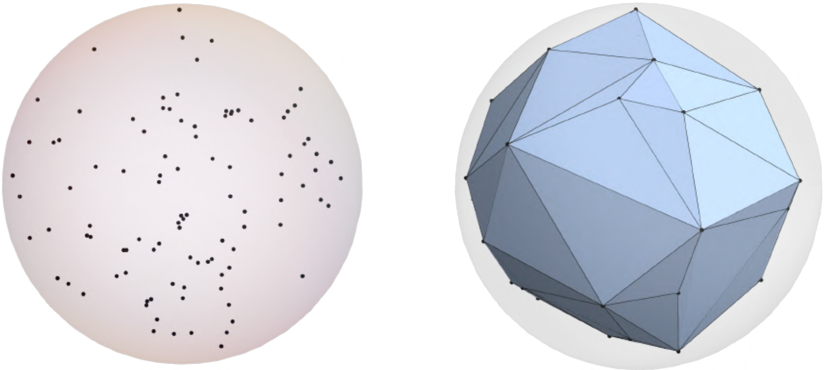

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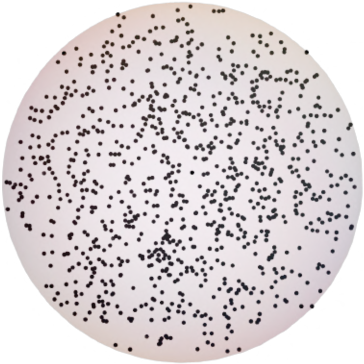

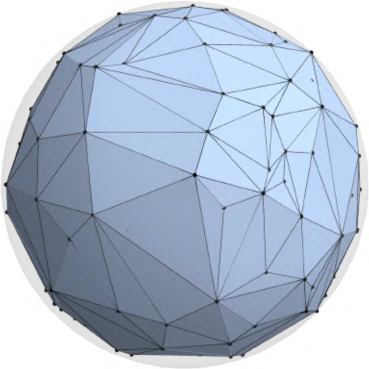

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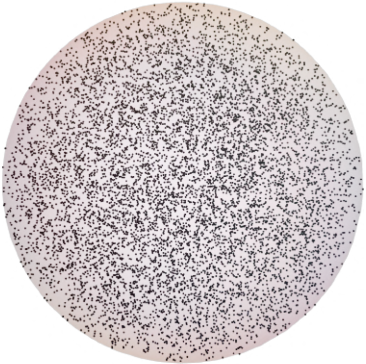

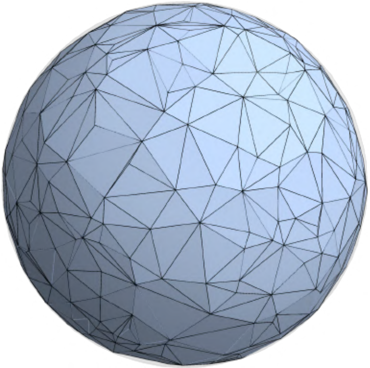

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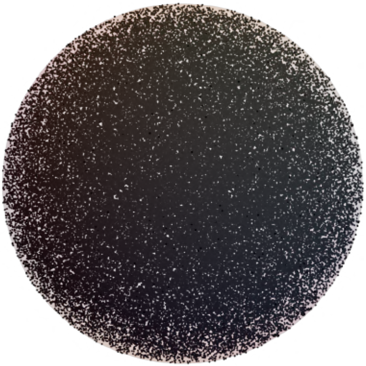

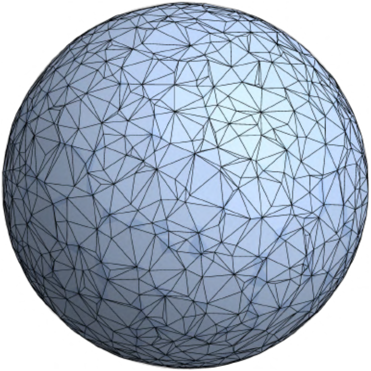

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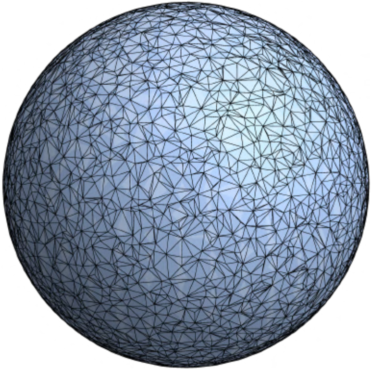

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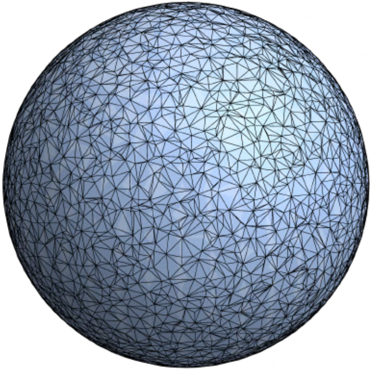

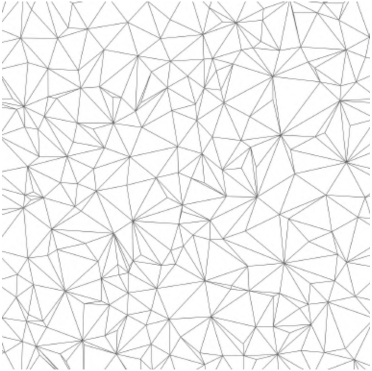

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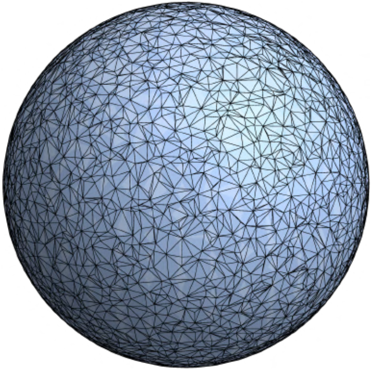

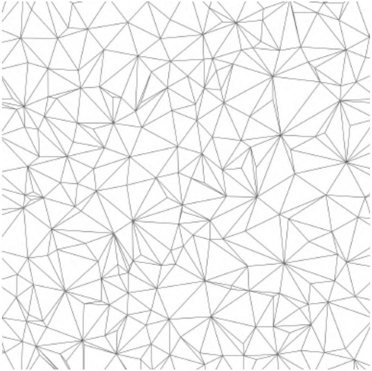

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Extreme value theory distributions (CDF is of the form $F(t) = \exp(-g(t))$)	Tessellation $(\mathcal{L}(f))^*$
Weibull: $g(t) = (-t)^\beta$, $t \le 0$	β-**Delaunay:** $f(h) = h^\beta$, $h \ge 0$
Fréchet: $g(t) = t^{-\beta}$, $t > 0$	β'-**Delaunay:** $f(h) = (-h)^{-\beta}$, $h < 0$
Gumbel: $g(t) = e^{-t}$	**Gaussian-Delaunay:** $f(h) = e^h$

- Given a stationary face-to-face random tessellation $\mathcal{T}$ denote by $X_{\mathcal{T},k}$ the stationary point process of k-dimensional faces of $\mathcal{T}$, e.g.

$$\mathcal{T}^{(k)} := \sum_{F \in \mathcal{F}_k(\mathcal{T})} \delta_{c(F)}, \quad 0 \le k \le d-1,$$

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Open question: Is there CLT for $\mathcal{D}'_{d,\beta}$ (especially when $\beta \approx d/2+1$)?

Thank you for attention!